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AFFINE CUBIC FUNCTIONS.

IV†. FUNCTIONS ON $\mathbb{C}^3$, NONSINGULAR AT INFINITY

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CONTENTS

† Parts I–III have appeared elsewhere (see References).

The object of this paper is to classify cubic functions f on $\mathbb{C}^3$ according to their singularities. A level surface of such a function extends to a cubic surface S in projective 3-space. The intersections S_∞, T_∞ of S and its Hessian quartic T with the plane at infinity are the same for all levels. We assume throughout that S_∞ is a non-singular cubic curve.

In §3 we show how the equisingularity class of T_∞ determines the number and multiplicities of critical points of f. In §2 we investigate $S_\infty \cap T_\infty$, and show that the equisingularity class of the pair (S_∞, T_∞) determines that of f. Next we study the case when some point of T_∞ has polar quadric a plane-pair; complete enumerations are given in §5 for the case when T_∞ contains a line, and in §6 for when it contains an Eckardt point of S.

In the final section we give a detailed analysis of cases when f has just two critical values, and show how to obtain a complete list of types of functions f.

0. Introduction

0.1. *Notation*

Although special notations will be used in some sections of the paper, the following will generally be used. We have a homogeneous cubic function F of four variables

$$F(x_0, x_1, x_2, x_3) = \sum_{i,j,k=0}^{3} a_{ijk} x_i x_j x_k,$$

where a_{ijk} is symmetric in its three suffices; we may also consider F as depending on the single vector $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$. Set

$$\partial_i F = \frac{1}{3}\frac{\partial F}{\partial x_i} = \sum_{j,k=0}^{3} a_{ijk} x_j x_k,$$

$$\partial_{ij} F = \frac{1}{6}\frac{\partial^2 F}{\partial x_i\, \partial x_j} = \sum_{k=0}^{3} a_{ijk} x_k,$$

and denote the completely polarized form by F^{p}:

$$F^{\mathrm{p}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i,j,k=0}^{3} a_{ijk} x_i y_j z_k.$$

The Hessian matrix $\theta(\boldsymbol{x})$ of F has the $\partial_{ij}F$ as entries; we denote its determinant by $H = H(F)$. If P is a point in projective space with coordinate vector $\boldsymbol{x}$, its polar quadric Σ_{P} is the locus of points $\boldsymbol{y}$ with $F^{\mathrm{p}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}) = 0$, and so has matrix $\theta(\boldsymbol{x})$. This may be singular: the locus of its singular points is called its vertex, and is the set of $\boldsymbol{y}$ with $F^{\mathrm{p}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0$ for all $\boldsymbol{z}$. This gives a symmetric condition on $\boldsymbol{x}$ and $\boldsymbol{y}$.

To consider the affine case we normally take $x_0 = 0$ as the plane at infinity, and set

$$f(x_1, x_2, x_3) = F(1, x_1, x_2, x_3), \quad F_\infty(x_1, x_2, x_3) = F(0, x_1, x_2, x_3).$$

Write $S = \{\boldsymbol{x}:F(\boldsymbol{x}) = 0\}$, $T = \{\boldsymbol{x}:H(F)(\boldsymbol{x}) = 0\}$ for the cubic surface and its Hessian quartic; S_∞ and T_∞ for their respective intersections with the plane $x_0 = 0$ at infinity.

We suppose throughout that S_∞ is a nonsingular cubic curve.

0.2. *Description of results*

The first main conclusion (§3) is that the equisingularity class of T_∞ (together with an assignation of a finite-valued invariant at each of its singular points) determines the number and multiplicities of critical points of f. In fact each critical point P of multiplicity exceeding one is a binode (or unode) of the corresponding level surface $f(\boldsymbol{x}) = a$. In the binode case, its polar quadric is a plane-pair, and the line of intersection of the planes meets the plane at infinity in a point L which we call *V-related* to P. Then L is a double point of T_∞; its type determines those of the singular points V-related to it.

In §2 we make a similar investigation for F with two critical points P and Q with $F(\mathrm{P}) = F(\mathrm{Q})$. If the line PQ meets the plane at infinity in M, then M is a point of $S_\infty \cap T_\infty$ of multiplicity exceeding one. An examination of all such multiple intersections leads to the conclusion that the equisingularity class of the pair (S_∞, T_∞) determines that of the function F.

This conclusion does not lead to a classification, however, as T_∞ is not independent of S_∞ and we cannot characterize geometrically just which pairs occur. Thus we next (§4) survey the case when there is a point L at infinity (necessarily singular on T_∞) whose polar quadric with respect to S is a plane-pair. We classify these into six types, of which the first two are not very significant (L is a node or cusp on T_∞). For the rest, complete enumerations are given in §5, which studies the case when T_∞ contains a line of points L whose polar quadrics have a common vertex P, and in §6, which deals with the case when S has an Eckardt point at infinity. In each of these cases, we are able to make a reduction to a problem of enumerating cubic functions on $\mathbb{C}^2$ with certain additional data. The completeness of the treatment allows us to answer subsidiary questions: the study of all cases when T_∞ contains a line; the overlap of the above two cases; cases when some polar quadric is a repeated plane, or when F has a critical point of corank 2, or when there is more than one Eckardt point at infinity.

We return to more general questions in §7, which is mainly devoted to obtaining a complete enumeration of functions f with just two critical values. There are 24 cases, in each of which f is essentially unique. In a final section we show that every combination of singularities that can arise occurs in the unfolding of one (at least) of these 24.

0.3. *Relation to singularity theory*

The motivation for writing this paper was to study the possible decompositions (in the sense of Lyashko (1976)) of the simple-elliptic singularity $\tilde{\mathrm{E}}_6$. Since this singularity has normal form

$$x^3+y^3+z^3+3\lambda xyz \quad (\lambda^3 \neq -1)$$

and its versal unfolding involves adding terms of lower degree, all these unfoldings are affine cubic functions satisfying our nonsingularity condition. Thus the final result above determines the possible decompositions of $\tilde{E}_6$.

There is particular interest in determining explicitly (in low codimensions) the canonical stratification of functions defined by Looijenga (1974) since one can use this to generalize to arbitrary dimensions results found by inspection for elementary catastrophes. If M^m is a compact manifold, and $f: M \to \mathbb{R}$ has only simple singularities, then the stratum of f has codimension $\Sigma\,\mu_P - \nu$, where the μ_P are the Milnor numbers of critical points P of f and ν is the number of critical levels. For $\tilde{E}_6$, though $\mu = 8$, as we have a one-parameter family we expect codimension 6.

Consider the versal unfolding of $\tilde{E}_6$, which has seven-dimensional parameter space:

$$f(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \lambda)(x, y, z) = x^3+y^3+z^3+3\lambda xyz+3\alpha yz+3\beta zx+3\gamma xy+3\delta x+3\epsilon y+3\zeta z.$$

Here, $\tilde{E}_6$ corresponds to the λ-axis. Each function has total multiplicity of critical points equal to 8, so the corresponding stratum has dimension $\nu - 1$ (if singularities are simple). In particular, the case $\nu = 2$ of functions with two critical values gives one-dimensional strata, each of which must be an orbit of the group action

$$(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \lambda)\,.\,t = (\alpha t, \beta t, \gamma t, \delta t^2, \epsilon t^2, \zeta t^2, \lambda),$$

and so is a parabola (or line) meeting the $\tilde{E}_6$ stratum at a unique point. At such points, the frontier condition breaks down, unless we segregate them into distinct strata.

These exceptional points are determined in §7.5, by calculating the j-invariant

$$j = \frac{(8\lambda - \lambda^4)^3}{64(1+\lambda^3)^2}$$

in each case. As well as the values $j = 0, 1$ there are fifteen other exceptional values of j. This makes a precise description of Looijenga's canonical stratification, even for this first nontrivial case, seem very difficult.

1. Review of geometry of cubic surfaces

1.1. *Plane sections*

The best known feature of the geometry of a nonsingular cubic surface S – the existence on it of 27 lines – will not play much part in this paper. We begin instead by considering plane sections. These are cubic curves, nonsingular unless we have a tangent plane π, when $\pi \cap S$ is usually a nodal cubic. If it is a cuspidal cubic, the point of contact is said to be a parabolic point of S. These points lie on the curve of intersection of S with its Hessian surface T. Since T is quartic, the parabolic curve $S \cap T$ is of degree 12.

If π passes through a line l of S, then $\pi \cap S$ will usually be a conic with chord l, meeting the conic at two points where π is tangent to S. If π is tangent to S at a parabolic point on l (there are in general two such), we have a conic with tangent. If π passes through one of the (ten) other lines on S meeting l, $\pi \cap S$ is a triangle, and π a tritangent plane. Exceptionally, the three lines may concur: in this case, the point of contact of π is an *Eckardt point* (E-point).

If S is singular, as well as the above possibilities $\pi \cap S$ may contain a repeated line or even a three-fold line. We call a non-singular point P of S an E-*point* if it is a triple point of the intersection $\pi \cap S$ (π the tangent plane to S at P). The parabolic curve is nonsingular except at singular points and E-points of S (though it may have repeated components: see below).

1.2. *Polar quadrics*

We may polarize the cubic form defining S, and thus associate to each point Q of projective 3-space its polar quadric Σ_Q with respect to S. For general Q, Σ_Q is nonsingular: the points Q for which it is singular are precisely the points of the Hessian surface T. The vertex of Σ_Q is also a point R of T, and Σ_R has vertex Q: we thus have an involution of T. Moreover Q lies on Σ_Q if $Q \in S$, and Q is a vertex of Σ_Q only if Q is a singular point of S.

At a singular point P, S has tangent cone Σ_P: P is said to be a conic node, or have type A_1, if Σ_P is a cone; a binode if Σ_P is a pair of planes; and a unode if Σ_P is a repeated plane. We shall normally exclude unodes (where F has a singular point of corank 2) from consideration, and will always exclude cases where S has a non-isolated singular point. If P is a binode, we call the intersection of the two planes the *pinch-line* of S at P.

There are in general just ten points $Q \in T$ with Σ_Q a plane-pair; such points are singular on T. The list of cases when Σ_Q is reducible is as follows:

Σ_Q a repeated plane through Q, arising iff S has a unode at Q;

Σ_Q a repeated plane not through Q, arising iff S is a cyclic cubic surface (e.g. $F = x_0^3 + \phi(x_1, x_2, x_3)$) with Q a vertex ($X_0$); here the plane of Σ_Q is a component of T;

Σ_Q a plane-pair with both planes through Q, arising iff S has a binode at Q;

Σ_Q a plane-pair with one plane through Q, arising iff S has an E-point at Q;

Σ_Q a plane-pair with neither plane through Q, the general case.

1.3. *Singularities*

The isolated singularities that a cubic surface (other than a cubic cone) can possess are all rational double points: we shall use the notation of Arnol'd (1972) for them. The types that actually occur are A_n $(1 \leqslant n \leqslant 5)$, D_4, D_5 and E_6. These are characterized geometrically as follows (see Bruce & Wall 1979).

If P is a conic node, it is of type A_1.

If P is a binode, we can take coordinates with P at X_0 and Σ_P as $x_1 x_2$: then

$$F = 6x_0 x_1 x_2 + \phi(x_1, x_2, x_3).$$

If the plane cubic $\phi = 0$ does not pass through the point Q ($x_1 = x_2 = 0$), P has type A_2. If it does, but neither $x_1 = 0$ nor $x_2 = 0$ is tangent at Q, P has type A_3. If (say) $x_1 = 0$ is an ordinary tangent (resp. inflexional tangent) to $\phi = 0$ at Q, then P has type A_4 (resp. A_5). Although $\phi = 0$ may be singular, Q cannot be a singular point else $x_1 = x_2 = 0$ would be a singular line on S. Observe that the pinch-line at a binode of type A_n lies on S only if $n \geqslant 3$.

If P is a unode, the plane π of Σ_P meets S in three lines through P. If these are distinct, P has type D_4; if two coincide, it has type D_5; and if all three coalesce, it has type E_6. Each of these lines may be called a pinch-line at the unode.

Surfaces may be classified according to the types of singularities they contain: for cubic surfaces the enumeration is due to Schläfli (1864); a modern proof is given in Bruce & Wall

(1979). With the assumption of isolated singularities and no triple point, the list of types is as follows:

nonsingular,

A_1, A_1^2, A_2, A_1^3, A_1A_2, A_3, A_1^4, $A_1^2A_2$, A_1A_3, A_2^2, A_4, D_4, $A_1^2A_3$, $A_1A_2^2$, A_1A_4, A_5, D_5, A_2^3, A_1A_5, E_6.

Here, for example, $A_1^2A_2$ denotes a surface with two singular points each of type A_1, and a third of type A_2. This classification takes no account of the presence or otherwise of E-points.

1.4. E-*lines on singular surfaces*

As E-points will play a major role in this paper, we now explore their relation with singular points. The key observation is

Proposition 1.4.1. *Let l be a line on S, that is not a double line. Then the following conditions are equivalent:*

(i) *The tangent planes to S at all regular points on l coincide.*

(ii) *There is a plane section of S of the type l^2m.*

(iii) *The line lies on the Hessian surface T.*

(iv) *Either l passes through two singular points of S, or it is a pinch line at a binode or unode of S.*

Proof. Take l as the line $x_2 = x_3 = 0$. The tangent plane at $P \equiv (p, q, 0, 0)$ is then

$$(a_{002}p^2+2a_{012}pq+a_{112}q^2)\,x_2+(a_{003}p^2+2a_{013}pq+a_{113}q^2)\,x_3 = 0,$$

so (i) and (ii) are each equivalent to the matrix

$$L = \begin{bmatrix} a_{002} & a_{012} & a_{112} \\ a_{003} & a_{013} & a_{113} \end{bmatrix}$$

having rank 1. Now the polar quadric Σ_P has matrix $\begin{bmatrix} 0 & A \\ A^T & B \end{bmatrix}$, where

$$A = \begin{bmatrix} a_{002}p+a_{012}q & a_{003}p+a_{013}q \\ a_{012}p+a_{112}q & a_{013}p+a_{113}q \end{bmatrix},$$

so the quadric is singular iff

$$0 = \det A = p^2(a_{002}a_{013}-a_{012}a_{003})+pq(a_{002}a_{113}-a_{112}a_{003})+q^2(a_{012}a_{113}-a_{112}a_{013}).$$

This is the condition that $(p, q, 0, 0) \in T$. It holds for all p, q again iff rank $P = 1$.

Finally, $(p, q, 0, 0)$ is singular iff the equation of the tangent plane vanishes identically. Now if $a_{002}p^2+2a_{012}pq+a_{112}q^2 = 0$ has distinct roots, then $a_{003}p^2+2a_{013}pq+a_{113}q^2$ is proportional to it iff there are two common roots, giving two singular points on l. If the roots are coincident – say $a_{002} = a_{012} = 0$ – then proportionality demands the vanishing of a_{003} and a_{013}. But $a_{003} = 0$ means that X_0 is a singular point, and $a_{013} = 0$ that its tangent cone reduces to $a_{022}x_2^2+2a_{023}x_2x_3+a_{033}x_3^2$, either a plane-pair with axis l (hence a pinch-line at a binode) or a repeated plane through l (hence X_0 a unode).

In proposition 1.4.1, condition (ii) also gives rise to a trifurcation: we may have $l \neq m$ and the intersection point of l and m (a) an E-point or (a′) a singular point, or we may have (b) $l = m$ (plane section l^3). In case (b), all regular points of l are E-points.

A routine calculation now shows that these cases occur when the singular points in question are of the following respective types:

(a) A_1 and A_1; A_3, D_4 or D_5^1

(a′) A_1 and A_n $(n > 1)$; A_4, D_5^2

(b) A_2 and A_2; A_5, E_6.

Here the notation D_5^1 (resp. D_5^2) refers to the line of multiplicity 1 (resp. 2) in the intersection $\pi \cap S$. A reference to the above classification shows that this list is complete.

It follows from §1.2 that for a regular point P on an E-line l, Σ_P is a cone unless P is an E-point when we have a plane-pair. Now for any point Q, if Σ_Q is a plane-pair, Q is a singular point of T. The converse holds except on pinch-lines:

Proposition 1.4.2. *Suppose* P *is a singular point of* T, *and* Σ_P *a cone, vertex* Q. *Then* Q *is a binode (or unode) of* S, *and* P *is on a pinch-line at* Q.

Proof. We can take P as X_0, Q as X_1. Since Σ_P has vertex Q, $a_{001} = a_{011} = a_{012} = a_{013} = 0$; since it is a cone,

$$D = \begin{vmatrix} a_{000} & a_{002} & a_{003} \\ a_{002} & a_{022} & a_{023} \\ a_{003} & a_{023} & a_{033} \end{vmatrix} \neq 0.$$

The coefficient of x_0^3 in H is

$$D(a_{111}x_1 + a_{112}x_2 + a_{113}x_3);$$

since this vanishes, $a_{111} = a_{112} = a_{113} = 0$. But now $Q \equiv X_1$ is seen to be singular on S, and X_0 is a vertex of Σ_Q, and hence lies on a pinch-line at Q.

The pinch-line at an A_2-singularity is not an E-line, since it does not lie on S. Apart from the singular point, it contains two other points (possibly coincident) whose polar quadrics are plane-pairs.

For a general point P on an E-line l of type (a) or (b′), Σ_P is a cone whose vertex Q also lies on l; if l is of type (b), the vertex of Σ_P meets l in a point Q. If l contains two singular points A, B then P and Q are mates in an involution with fixed points A, B. If l is a pinch-line at C, we always have $Q \equiv C$.

Any E-line is multiply contained in the parabolic curve; we now give these multiplicities, which will play an important part below. We also give the multiplicities as lines on S (these must sum to 27) which are more readily obtained.

Proposition 1.4.3. *These multiplicities are given by table* 1.4.3.

Table 1.4.3

multiplicity	parabolic curve	line on S
line through points of types A_r, A_s	$r+s$	$(r+1)(s+1)$
pinch-line at binode type A_r	$2(r-2)$	$\frac{1}{2}r(r+1)$
pinch-line at unode type D_4	2	8
pinch-line at unode type D_5^1	2	10
pinch-line at unode type D_5^2	6	16
pinch-line at unode type E_6	10	27

A line through a singular point of type A_r that is not an E-line does not lie on the parabolic curve; it has multiplicity $r+1$ as a line on S.

The cases of a line through two A_1 points or an A_1 and an A_2, or the pinch line at an A_3, will be verified in §§2.2 and 2.3 respectively. The remainder concern surfaces each of which (except A_2^3) has a unique normal form. The parabolic curve can be determined explicitly in these cases by a direct calculation, and the result thus verified.

2. Critical points at the same level

2.1. *Relation to points of $S_\infty \cap T_\infty$*

If f has distinct critical points P, Q at the same level, $f(\mathrm{P}) = f(\mathrm{Q}) = a$, then the line PQ is an E-line of the level surface S, $f(\boldsymbol{x}) = a$, and so is a multiple intersection of S and T. Thus the point where it meets $x_0 = 0$ is a point of multiple intersection of S_∞ and T_∞.

Conversely, consider any such point of multiple intersection. We may choose coordinates with the point at X_1 and the tangent plane there to S as $x_2 = 0$. This plane meets S where

$$0 = 3x_1(a_{001}x_0^2 + 2a_{013}x_0x_3 + a_{113}x_3^2) + a_{000}x_0^3 + 3a_{003}x_0^2x_3 + 3a_{033}x_0x_3^2 + a_{333}x_3^3.$$

Since X_1 is a parabolic point, this curve has a cusp at X_1, so $a_{013}^2 = a_{001}a_{133}$.

If $a_{001} = a_{013} = a_{133} = 0$, X_1 is an E-point of all level surfaces: such a point we call an E_∞-*point* of f. Otherwise we can choose the cuspidal tangent to lie along $x_3 = 0$ (so that $a_{001} = a_{013} = 0$) unless it lies along $x_0 = 0$ (the case when $a_{013} = a_{133} = 0$).

Since X_1 is a point of multiple intersection of S_∞ and T_∞, the coefficient of $x_1^3x_3$ in H must vanish. We calculate this coefficient as

$$-a_{003}a_{112}^2a_{133} \quad (\text{if } a_{001} = a_{013} = 0),$$

$$a_{001}a_{112}^2a_{333} \quad (\text{if } a_{013} = a_{133} = 0).$$

Now a_{112} cannot vanish, as X_1 is not singular on S_∞, nor can we have $a_{133} = a_{333} = 0$, else $x_2 = 0$ would be a component of S_∞, again imposing a singularity. Thus if X_1 is not an E_∞-point, the second of the above two cases cannot occur, and we must have $a_{003} = 0$.

But now the level surface for which $a_{000} = 0$ meets $x_2 = 0$ in

$$0 = x_3^2(3a_{033}x_0 + 3a_{133}x_1 + a_{333}x_3),$$

so that $x_2 = x_3 = 0$ is an E-line on this surface. We have thus proved

Proposition 2.1.1. *A point on S is a point of multiple intersection of S_∞ and T_∞ if and only if it is either an E_∞-point or lies on an E-line of a (unique) level surface of f.*

By proposition 1.4.3 such a line either passes through two critical points of f (at the same level) or is a pinch-line at a binode (or unode) which is thus V-related to our point. We examine these cases in turn.

2.2. *Distinct critical points*

We continue with coordinates as in §2.1: thus the tangent plane $x_2 = 0$ at X_1 meets S twice in the E-line $x_2 = x_3 = 0$. The tangent plane at a point $\mathrm{P} \equiv (p, q, 0, 0)$ of the line is thus

$$0 = x_2(a_{002}p^2 + 2a_{012}pq + a_{112}q^2),$$

and P is singular if the expression in parentheses vanishes. If $a_{012}^2 \neq a_{002}a_{112}$ the line contains two distinct critical points. In this case, X_1 is not a singular point of T_∞; we wish to check the order of contact of S_∞ and T_∞ at X_1.

We can set $x_1 = 1$ for this: then if x_3 has order t along either curve, x_2 has order t^2, so the terms of least order are $3(a_{112}x_1^2x_2 + a_{133}x_1x_3^2)$ for F_∞ and the terms in $x_1^3x_2$, $x_1^2x_3^2$ for H_∞. A direct calculation from the determinant shows that these terms are

$$a_{112}a_{133}(a_{012}^2 - a_{002}a_{112})\,x_1^3x_2 + (a_{012}a_{133} - a_{033}a_{112})^2x_1^2x_3^2.$$

Thus the intersection number is at least 3 if and only if

$$(a_{012}a_{133} - a_{033}a_{112})^2 = a_{133}^2(a_{012}^2 - a_{002}a_{112}),$$

i.e.
$$a_{112}(a_{002}a_{133}^2 - 2a_{012}a_{033}a_{133} + a_{112}a_{033}^2) = 0,$$

which means that $(-a_{033}/a_{133},\ 0,\ 0)$ is one of the two critical points under study.

On the other hand if $(\alpha, 0, 0)$ and $(\beta, 0, 0)$ are these critical points, so that

$$a_{012} = -\tfrac{1}{2}(\alpha+\beta)\,a_{112} \quad a_{002} = \alpha\beta a_{112},$$

then
$$\partial_{ij}f(\alpha, 0, 0) = a_{0ij} + \alpha a_{1ij}$$

vanishes for $(i,j) = (1, 1)$ or $(1, 3)$, so the determinant of this matrix is $-(\partial_{12}F)^2\partial_{33}F$, which equals

$$-(a_{012} + \alpha a_{112})^2(a_{033} + \alpha a_{133}) = -\tfrac{1}{4}(\beta - \alpha)^2a_{112}(a_{033} + \alpha a_{133}),$$

which vanishes only when $\alpha = -a_{033}/a_{133}$. Hence this is the condition for the critical point to have type higher than A_1.

Thus only one point Q on the line can have higher type than A_1 (indeed, the line joining two A_2 consists of E-points), and it does so iff the intersection multiplicity of S_∞ and T_∞ at X_1 exceeds 2.

Now substitution of $x_2 = 0$ in H_∞, reduces this to

$$\begin{vmatrix} a_{033}x_3 & a_{012}x_1 + a_{023}x_3 \\ a_{133}x_3 & a_{112}x_1 + a_{123}x_3 \end{vmatrix}^2,$$

so in fact x_2 is a bitangent, with second contact P where

$$0 = \begin{vmatrix} a_{012} & a_{033} \\ a_{112} & a_{133} \end{vmatrix} x_1 + \begin{vmatrix} a_{023} & a_{033} \\ a_{123} & a_{133} \end{vmatrix} x_3.$$

I claim that in the above situation, P is on the pinch line of Q. We can take Q at X_0: then $a_{033} = 0$, so $P = (a_{023}, 0, -a_{012})$ and the matrix of the polar quadric of P has vanishing first row, and hence indeed has X_0 as a vertex. Observe that P coincides with X_1 only when $a_{012} = 0$, so the two critical points coincide.

We also deduce that Q has type A_3 (or higher) $\Leftrightarrow P \in S_\infty$.

We conclude our analysis of this case by proving

Proposition 2.2.1 *If the critical points on X_0X_1 are of types A_1, A_n then the intersection multiplicity of S_∞, T_∞ at X_1 is $n+1$.*

Proof. Let us write m for the intersection multiplicity: then we showed above that $n \geqslant 2 \Leftrightarrow m \geqslant 3$. We now check the condition for $m \geqslant 4$. Near X_1 on S_∞, T_∞ we take $x_1 = 1$, $x_3 = t$: then x_2 has order t^2, and we have already checked the coefficient of t^2 in the two cases. Terms of order t^3 in the equation are $6a_{123}x_1x_2x_3 + a_{333}x_3^3$ (F_∞) and the terms in $x_1^2x_2x_3$, $x_1x_3^3$ (H_∞). We may continue to suppose $0 = a_{002} = a_{033}$: then $x_2 = -(a_{133}/a_{112})\,t^2 +$ higher-order terms, and the terms of degree 2 in

$$\Psi = 3H_\infty - a_{012}^2a_{133}F_\infty$$

cancel out. The coefficient of $x_1^2 x_2 x_3$ in H_∞ is

$$a_{012}^2(a_{112}a_{333}+2a_{123}a_{133})-2a_{012}a_{023}a_{112}a_{133},$$

and the coefficient of $x_1 x_3^3$ is that in

$$\begin{vmatrix} a_{012}x_1+a_{023}x_3 & 0 \\ a_{112}x_1+a_{123}x_3 & a_{133}x_3 \end{vmatrix}^2, \quad \text{namely} \quad 2a_{012}a_{023}a_{133}^2.$$

Hence the coefficient of t^3 in Ψ is

$$\begin{aligned} &3(a_{012}^2a_{112}a_{333}+2a_{012}^2a_{123}a_{133}-2a_{012}a_{023}a_{112}a_{133})(-a_{133}/a_{112}) \\ &\quad +6a_{012}a_{023}a_{133}^2-a_{012}^2a_{133}\{6a_{123}(-a_{133}/a_{112})+a_{333}\} \\ &\quad = 4a_{012}a_{133}(3a_{023}a_{133}-a_{012}a_{333}). \end{aligned}$$

This vanishes iff $m \geqslant 4$. On the other hand, $n \geqslant 3$ iff $P(a_{023}, 0, -a_{012})$ lies in S_∞, i.e.

$$0 = a_{012}^2(3a_{133}a_{023}-a_{333}a_{012}).$$

As $a_{133} \neq 0$, these conditions are equivalent.

The proof for the higher-order cases is now completed by referring to proposition 1.4.3: we showed there that for $n \geqslant 3$, the parabolic curve has the line X_0X_1 as a component with multiplicity $n+1$. As π_∞ passes through no singular points, it meets this line transversely at a point on no other component of the parabolic curve, so the local intersection number m of F_∞ and H_∞ must equal $n+1$.

Reversing this argument allows us to complete the proof of proposition 1.4.3 for the cases hitherto omitted.

2.3. *Coincident critical points*

This case is more easily handled by a different normalization of coordinates. Take the binode of S at X_0, and its polar quadric as x_2x_3: then

$$F = 6x_0x_2x_3+\phi(x_1, x_2, x_3),$$

where, since X_0 is an A_3 (at least), $a_{111} = 0$. The tangent plane along the pinch-line is

$$a_{112}x_2+a_{113}x_3 = 0.$$

The coefficient of x_1^2 in H is $\qquad (a_{112}x_2-a_{113}x_3)^2.$

Now a_{112} and a_{113} do not both vanish (X_1 is not singular on S_∞), so T has a double point at X_1 with tangents coincident along

$$a_{112}x_2-a_{113}x_3 = 0.$$

Unless this agrees with the tangent (above) to S_∞, the intersection number is 2: however, it only agrees if a_{112} or a_{113} vanishes, which is the condition for X_0 to be of type A_4 (at least).

PROPOSITION 2.3.1. *If* X_1 *lies on the pinch-line of a binode of type* A_n ($n = 3, 4, 5$), *the intersection multiplicity of* S_∞ *and* T *at* X_1 *is* $2(n-2)$.

As for proposition 2.2.1, the remaining cases are covered by an appeal to proposition 1.4.3.

2.4. *The* E_∞*-point case*

We return to the normalization of §2.1. The restriction of f to the plane $x_2 = 0$ now depends only on the affine coordinate x_3: write

$$\phi_l(x_3) = a_{333}x_3^3+3a_{033}x_3^2+3a_{003}x_3+a_{000},$$

where (as already noted) $a_{333} \neq 0$. We say the E-point X_1 is type a (resp. b) if the critical points of ϕ_l are distinct (resp. coincident), i.e. if $a_{333}a_{033} \neq$ (resp. $=$) a_{033}^2. For type a there are two choices of a_{000} (for type b, only one) for which ϕ_l has a repeated factor. Such a repeated factor yields an E-line of the corresponding level surface.

Moreover, for type a, each of the two level surfaces in question has an intersection with $x_2 = 0$ of the type l^2m. As the point X_1 of intersection of l and m is an E-point (not a singular point), these E-lines are both of type a in the terminology of § 1.4, so each one passes through critical points of types A_1^2, A_3, D_4 or D_5^1. In case b, the relevant level surface meets $x_2 = 0$ in a threefold line l^3, so l is an E-line of type b in the terminology of § 1.4: it thus passes through critical points of types A_2^2, A_5 or E_6.

The polar quadric of X_1 is a plane-pair with the tangent plane $x_2 = 0$ as one of the planes and the other plane not through X_1. We normalize coordinates further by taking this other plane as $x_1 = 0$. Then

$$F = 3x_1^2x_2 + \phi(x_0, x_2, x_3).$$

The line $x_2 = 0$ is an inflexional tangent at X_1 to S_∞, which is given by an equation almost in Weierstrass normal form. The critical points of f are given by the equations

$$0 = x_1x_2, \quad 0 = 3x_1^2 + \partial_2\phi, \quad 0 = \partial_3\phi.$$

On the plane $x_2 = 0$, $\partial_3\phi$ vanishes along the two E-lines (one for type b), and the second equation determines equal and opposite values of x_1 (in general distinct) on each. If we consider ϕ as a function on an affine plane, its restriction ϕ_l to $x_2 = 0$ has two critical points P, Q (which coincide for type b) and each of these yields two critical points of f, which coincide if $\partial_2\phi$ vanishes at P, Q respectively. We further classify the E_∞-point to have type a_r ($r = 0, 1, 2$) or b_r ($r = 0, 1$) if $\partial_2\phi$ vanishes at r of the points P, Q. Moreover, we call all these critical points on the tangent plane E-related to X_1.

The properties of this classification are summarized in

THEOREM 2.4.1. *An* E_∞-*point with no* E-*related critical point of corank* 2 *has properties given by table* 2.4.1.

TABLE 2.4.1

type of E_∞-point	a_0	a_1	a_2	b_0	b_1
E-related critical points of f	A_1^2/A_1^2	A_1^2/A_3	A_3/A_3	A_2^2	A_5
intersection number of S_∞ and T_∞	2	2	2	4	6

Proof. The second row in the table is given by the preceding discussion.

We choose *one* of the levels that gives us an E-line and apply proposition 1.4.3 (now proved in all cases) to obtain the intersection number of S_∞ and T_∞.

3. Nets of quadrics, and critical points

We consider the cubic function $f = f(x_1, x_2, x_3)$ of three variables, with our standard notation.

3.1. *Background: summary of Wall* (1978)

The critical points of f are given by the equations $0 = \partial f/\partial x_1 = \partial f/\partial x_2 = \partial f/\partial x_3$. Thus they satisfy $0 = \Sigma_{i=1}^3 y_i\, \partial f/\partial x_i$ for all (y_1, y_2, y_3). With $\mathbb{C}^3$ identified with the subset $x_0 \neq 0$ of $P_3(\mathbb{C})$ in the usual way, this means that x lies on the polar quadrics of all the points

$(0, y_1, y_2, y_3)$. Thus the critical points of f are the base points of the net of polar quadrics Σ_P with respect to S of all points P on $x_0 = 0$. If this net has a base point Q on the plane $x_0 = 0$, then since $0 = \partial F/\partial x_1 = \partial F/\partial x_2 = \partial F/\partial x_3$ at Q, Q is a singular point of the curve S_∞ of intersection of S with the plane $x_0 = 0$. We shall suppose throughout this paper that f is such that the curve S_∞ is nonsingular, so that none of the base points can lie on $x_0 = 0$.

The quadrics $0 = \Sigma_{i=1}^3 y_i \partial F/\partial x_i = F^P(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{y})$ $(y_0 = 0)$ of the net are parametrized by the points Q (with coordinates $\boldsymbol{y}$) on $y_0 = 0$. The quadric Σ_Q is singular iff Q lies on T. In Wall (1978) I classified nets of quadrics parametrized by $\boldsymbol{\lambda} = (\lambda, \mu, \nu)$ in terms of the quartic curve of points $\boldsymbol{\lambda}$ corresponding to singular quadrics. We thus see that in the present situation this discriminant curve coincides with the intersection T_∞ of T with $x_0 = 0$. We now apply the results of Wall (1978).

The discriminant curve T_∞ determines the net up to the action of $GL_4(\mathbb{C})$ and a 36-fold ambiguity. Part of this ambiguity is resolved by the study of base points of the adjugate system. This system is defined by halving the system cut on T_∞ by the cubic curves

$$(X_0, X_1, X_2, X_3) \operatorname{adj} \theta(\boldsymbol{y}) (X_0, X_1, X_2, X_3)^T = 0$$

for some $(X_0, X_1, X_2, X_3) \in \mathbb{C}^4$. These adjugate base points can occur only at singular points of T_∞; a singular (double) point of type A_n may have multiplicity r as adjugate base point for any r with $0 \leqslant r \leqslant \frac{1}{2}(n+1)$. Triple points may also occur. A singular point P of T_∞ is an adjugate base point (with multiplicity $r \geqslant 1$) iff Σ_P is a plane-pair (if Σ_P is a repeated plane, P is a triple point).

In Wall (1978) I classified nets as stable and unstable. The theorem in Wall (1980*a*, §5) is as follows.

THEOREM 3.1.1. *Let f be a cubic function on $\mathbb{C}^3$ such that (with the above notation) S_∞ is nonsingular. Then the following conditions are equivalent:*

(i) *f has a critical point of corank* 2; (ii) *the net of quadrics discussed above is unstable*; (iii) *the curve T_∞ has a repeated component.*

Moreover, the case when these conditions are satisfied was fully discussed in Wall (1980*a*). We can thus exclude it (except for occasional mention) in this paper. Now for stable nets I determined in Wall (1980*c*) the multiplicities of the base points; this result is given next.

3.2. *Summary of Wall* (1980*c*)

Write our net of quadrics as

$$F(\boldsymbol{\lambda}, \boldsymbol{x}) = \boldsymbol{x}^T(\lambda_0 M_0 + \lambda_1 M_1 + \lambda_2 M_2)\, \boldsymbol{x}$$

in matrix notation, with quadrics

$$Q_\lambda = \{\boldsymbol{x} : F(\boldsymbol{\lambda}, \boldsymbol{x}) = 0\}.$$

Define the total variety

$$V = \{(\boldsymbol{\lambda}, \boldsymbol{x}) : F(\boldsymbol{\lambda}, \boldsymbol{x}) = 0\},$$

the variety of base points

$$B = \{\boldsymbol{x} : (\forall \boldsymbol{\lambda}),\quad F(\boldsymbol{\lambda}, \boldsymbol{x}) = 0\},$$

and the discriminant

$$\Delta = \{\boldsymbol{\lambda} : \det(\lambda_0 M_0 + \lambda_1 M_1 + \lambda_2 M_2) = 0\}.$$

For any variety W, write $S(W)$ for its set of singular points.

These notations are taken from Wall (1980*c*), and all references in this section will be to that paper. Now (lemma 1.2) a point $\boldsymbol{x}$ of B belongs to $S(B)$ iff there exists $\boldsymbol{\lambda}$ (necessarily in Δ) with $x \in S(Q_{\boldsymbol{\lambda}})$: an equivalent condition is that $M_0\,\boldsymbol{x}$, $M_1\,\boldsymbol{x}$, $M_2\,\boldsymbol{x}$ are linearly dependent. If these span a two-dimensional vector space, so that the linear relation (hence the point $\boldsymbol{\lambda}$) is projectively unique, we call $\boldsymbol{x}$ *tame*. In this case, we say $\boldsymbol{x}$ is V-related to $\boldsymbol{\lambda}$. For a stable net (lemma 4.1), all base points are tame. Further (see §4.1) for a net of quadrics in $P_3(\mathbb{C})$, a tame point $\boldsymbol{x}$ has multiplicity $l+1$ as base point of the net iff it is a singular point of B of type A_l: thus in particular, $\boldsymbol{x}$ belongs to $S(B)$ iff the multiplicity is at least 2.

Now (lemma 1.1), $(\boldsymbol{\lambda}, \boldsymbol{x}) \in S(V)$ iff $\boldsymbol{x} \in S(Q_\lambda) \cap B$. Thus if all base points are tame, projection gives a bijection between $S(V)$ and $S(B)$. Moreover (proposition 1.3) if $\boldsymbol{x} \in S(B)$ is tame, and is V-related to $\boldsymbol{\lambda}$, the singular points $(\boldsymbol{\lambda}, \boldsymbol{x})$ of V and $\boldsymbol{x}$ of B have the same type. For our present purposes, this means type A_l for the same value of l.

The main results give relations between $S(\Delta)$ and $S(V)$, which are most conveniently expressed in table 3.2.1. In view of the results above we can equally well regard these as relations between $S(\Delta)$ and $S(B)$. Moreover, it will avoid confusion in later reference if we list the multiplicity $l+1$ as base point, rather than the type A_l of singularity of B. Note however that since $l \geqslant 1$, these multiplicities are all greater than or equal to 2: any entry in table 3.2.1 yielding multiplicity less than or equal to 1 has to be deleted.

Table 3.2.1

nature of singular point $\boldsymbol{\lambda}$ on Δ	corank of Q_λ	adjugate base-point multiplicity	multiplicities of V-related base-points	references to equations in Wall (1980*c*)
A_m	1	0	$m+1$	(1.4)
A_m	2	$1 \leqslant s \leqslant \frac{1}{2}(m+1)$	$s,\ m-s+1$	(3.1)–(3.3)
D_{m+1}	2	—	$m+1$	(3.4), (4.2)
D_4	3	—	2, 2, 2, 2	(3.6)
D_5	3	—	4, 2, 2	(3.6)
D_6	3	—	4, 4	(3.6)
E_6	3	—	6, 2	(3.6)
E_7	3	—	8	(3.6)

No other cases arise for stable nets. The first row in the table can be regarded as the case $s = 0$ of the second.

3.3. *Special character of the net*

The relations described in §3.2 are valid for all (stable) nets. The nets occurring here are not, however, arbitrary. It was shown by Turnbull (1935) (I am indebted to the referee for this reference) that a net occurs as net of polar quadrics of a cubic iff it satisfies a certain condition – and then it so occurs in ∞^2 ways. This condition, which also appears in work of Barth (1977), may be stated as follows. If M_0, M_1, M_2 are matrices of independent quadrics of the net, with M_0 nonsingular, $M_1 M_0^{-1} M_2 - M_2 M_0^{-1} M_1$ is singular.

The condition in this form is not directly relevant to our classification, but we can obtain strong restrictions by a geometric analysis of the V-relation. We assume that f has no critical points of corank 2, so (theorem 3.1.1) the net is stable.

As in §3.1, we have $\Delta = T_\infty$, and for any $P \in \pi_\infty$, the quadric Q_λ is the polar quadric of P with respect to S. A base point X of the net is a singular point of f, of type A_k for some k (the corank is 1). Its multiplicity as base point is the local intersection number of the quadrics

$\partial f/\partial x_i = 0$ ($i = 1, 2, 3$), which coincides with the Milnor number k. Thus X is a binode of the corresponding level surface $S \Leftrightarrow k \geqslant 2 \Leftrightarrow$ X is V-related to some point P of T_∞, i.e. $X \in S(Q_\lambda)$. But then (by §0.1) P is on the vertex of the polar quadric of X.

This confirms that if $k = 1$, X (a conic node) is not V-related to anything. For $k \geqslant 2$, X is a binode; the vertex of its polar quadric is its pinch-line, and X is V-related to the unique point P where this line meets π_∞. It follows that $P \in S \Leftrightarrow k \geqslant 3$, for the pinch-line of an A_2-point meets the surface only in that point.

Thus suppose $P \in T_\infty$ does not lie on S_∞. Then all V-related critical points are of type A_2. This excludes all but five of the cases in table 3.2.1.

Next suppose P is a singular point of T_∞ that does lie on S_∞. Then it is a point of multiple intersection of S_∞ and T_∞, so we can apply the results of §2. If P is not an E_∞-point, its polar quadric is a cone: thus P is of type A_l (some l); there is a unique V-related X, which has type A_{l+1}, so $l = 2$ or 3 (if $l = 4$, P is an E_∞-point). For P an E_∞-point, we have five cases, as listed in table 2.4.1: the V-related critical points here are those on the line $x_1 = x_2 = 0$ in the notation of §2.4.

Thus of the 23 possible types of singular point of Δ, listed in Wall (1978), only nine occur in the present problem, namely those in table 3.3.1.

TABLE 3.3.1

type of case	singularity of T_∞	corank of Q_λ	adjugate base-point multiplicity	multiplicity of V-related base-points
not on S_∞	A_1	1	0	2
—	A_1	2	1	—
—	A_2	2	1	2
—	A_3	2	2	2, 2
—	D_4	3	—	2, 2, 2, 2
on S_∞	A_2	1	0	3
—	A_3	1	0	4
E_∞, a_0	A_1	2	1	—
E_∞, a_1	A_3	2	1	3
E_∞, a_2	A_5	2	3	3, 3
E_∞, b_0	A_1	2	1	—
E_∞, b_1	A_5	2	1	5

In §4 we shall discuss all cases when T_∞ contains a point P whose polar quadric has corank greater than or equal to 2. In §6 we enumerate all cases when there is an E_∞-point, and in §5 all cases when T_∞ contains a line: it is shown in §4.2 that this includes all cases of the fourth and fifth rows above. The V-relation in these cases is discussed in §5.5 and §6.3: we shall see that all the cases just listed do occur.

Just as we do not obtain all nets, we do not need the full classification of the net corresponding to f to determine the critical points of f. In particular, a singular point P of type A_1 of T_∞, with polar quadric a plane-pair, has no V-related base points and will not affect our classification (unless it also lies on S_∞ and is thus an E_∞-point).

4. Enumerations (general considerations)

4.1. *List of six cases*

The number of cases arising is so large that we have not yet been able to give a complete list. Under any one of several auxiliary conditions, however, this becomes feasible. Most of these conditions involve having a point P at infinity whose polar quadric Σ_P is reducible (so P is an adjugate base point for the net). Indeed, the results of §2 cannot easily be applied, since the calculations required to determine contacts of S_∞ and T_∞ are so complicated in general.

In this section we give a general discussion of cases arising when Σ_P is reducible: particular cases will be treated in more detail in later sections. We start from the projective (rather than affine) viewpoint. Let $\Sigma_P = \pi_1\pi_2$. Then if $P \in \pi_1 \cap \pi_2$, P is a singular point of S (hence of S_∞), so we exclude this case. Otherwise we may take projective coordinates (w, x, y, z) with P at W, $\pi_1 \cap \pi_2$ as $w = x = 0$.

Take $\pi_1\pi_2 = aw^2+2bwx+cx^2$. Then

$$F = aw^3+3bw^2x+3cwx^2+\phi(x, y, z),$$

$$H = \det\begin{bmatrix} aw+bx & bw+cx & 0 & 0 \\ bw+cx & cw+\partial_{11}\phi & \partial_{12}\phi & \partial_{13}\phi \\ 0 & \partial_{12}\phi & \partial_{22}\phi & \partial_{23}\phi \\ 0 & \partial_{13}\phi & \partial_{23}\phi & \partial_{33}\phi \end{bmatrix}$$

$$= (aw+bx)\,H(\phi)+\{(aw+bx)\,cw-(bw+cx)^2\}\,M(\phi),$$

where $H(\phi)$ is the Hessian of ϕ, and

$$M(\phi) = \partial_{22}\phi\,\partial_{33}\phi-(\partial_{23}\phi)^2.$$

The equation of the tangent cone to T at P is thus

$$(ac-b^2)\,M(\phi) = 0:$$

it is a quadric cone, which is of rank 3 (irreducible) iff $ac \neq b^2$ and $\partial_{22}\phi$, $\partial_{23}\phi$ and $\partial_{33}\phi$ are linearly independent.

Now choose a plane π_∞ through P and consider the traces of all the above on π_∞. By considering the tangent cone $(ac-b^2)\,M_\infty(\phi)$ to T_∞, we may distinguish six cases, as follows. In the first five, $ac \neq b^2$.

(i) $M_\infty(\phi)$ has distinct factors; P has type A_1 on T_∞.

Next suppose $M_\infty = l^2$ a repeated line, and consider the coefficient $aH_\infty(\phi)-bcxM_\infty(\phi)$ of w in H_∞.

(ii) The line does not divide this, so P has type A_2 on T_∞.

(iii) The parameter $a = 0$, so $P \in S$ is an E_∞-point.

(iv) The line l divides H_∞.

If finally P is a triple point of T_∞, then either

(v) $M_\infty(\phi)$ vanishes identically, or

(vi) $ac = b^2$, so $\pi_1 = \pi_2$.

4.2. *Preliminary discussion*

We now discuss these six cases. In (i), there are no critical points V-related to P, so this condition is not relevant for our classification. In (ii), P had type A_2 and so (by §2.2) is V-related to a single critical point of multiplicity 2. This case is still too general for a full discussion. Some discussion of cases (i) and (ii) is given in §4.3.

Case (iii) will be fully discussed in §6. For case (iv) we have

LEMMA 4.2.1. *If l is a line of intersection of $H(\phi) = 0$ and $M(\phi) = 0$, then the polar quadrics of points of l form a pencil with a common vertex.*

The case when π_∞ contains a line l whose polar quadrics Σ_P ($P \in l$) form a pencil with a common vertex will be fully discussed in §5. Case (v) is dealt with by

LEMMA 4.2.2. *If $M_\infty(\phi)$ vanishes identically, the corresponding affine function f has a critical point of corank* 2.

Thus case (v) has been discussed in Wall (1980*a*). Finally, in case (vi) Σ_P is a repeated plane π^2. If π meets the plane at infinity in the line l, then for each $Q \in l$, Σ_Q has P as a vertex. Thus here again the net contains a pencil with common vertex and we refer to §5.

For the proof of the lemmas, note that if $M(\phi)$ vanishes on some linear subspace, then

$$\partial_{22}\phi = \alpha^2 k, \quad \partial_{23}\phi = \alpha\beta k, \quad \partial_{33}\phi = \beta^2 k$$

for some constants α, β and linear form k on the same subspace. Substituting, we have

$$H(\phi) = -k(\alpha\,\partial_{13}\phi - \beta\,\partial_{12}\phi)^2$$

on the same subspace.

Proof of lemma 4.2.1. Here we have a line l, and so a subspace of points $(x_0, \lambda y_1, \lambda y_2, \lambda y_3)$ with y_1, y_2, y_3 fixed and x_0, λ variable: we can take $k = \lambda$. As $H(\phi)$ vanishes on l, $\alpha\,\partial_{13}\phi = \beta\,\partial_{12}\phi$. If $(\alpha, \beta) \neq (0, 0)$, the polar quadrics all have vertex $(0, 0, -\beta, \alpha)$: when $\alpha = \beta = 0$, they have vertex $(0, 0, \partial_{13}\phi, -\partial_{12}\phi)$ (where these are evaluated at (y_1, y_2, y_3)).

Proof of lemma 4.2.2. Here the subspace is π_∞. We have

$$M_\infty(\phi) = 0, \quad H_\infty(\phi) = -k(\alpha\,\partial_{13}\phi - \beta\,\partial_{12}\phi)^2,$$

so $H_\infty = (ax_0 + bx_1)\,H_\infty(\phi)$ has a repeated factor. The conclusion now follows by theorem 3.1.1.

4.3. *Remarks on cases* (i) *and* (ii)

We return to the general case of (i) and (ii) above. The case when the three planes π_1, π_2 and π_∞ are collinear (i.e. π_1 and π_2 are parallel) will be fully discussed in §5. Otherwise we can take $\pi_\infty = x_0$, P as X_1, $\pi_1\pi_2 = x_1^2 - x_2^2$ and so

$$F = x_1^3 - 3x_1x_2^2 + \phi(x_0, x_2, x_3).$$

The critical points of f lie on π_1 or π_2 and are also critical for $f \mid \pi_i$; conversely the critical points of $f \mid \pi_i$ are also critical for f. In case (i) there are no critical points on $\pi_1 \cap \pi_2$; in case (ii), there is just one, of multiplicity 2: each $f \mid \pi_i$ has a critical point of multiplicity 1 there.

As in §4.1,

$$H(F) = x_1 H(\phi) - (x_1^2 + x_2^2)\,M(\phi),$$

where $H(\phi)$ is the Hessian of ϕ and

$$M(\phi) = \partial_{00}\phi\,\partial_{33}\phi - (\partial_{03}\phi)^2.$$

T_∞ has a double point at X_1 (of type A_1 for (i), A_2 for (ii)), but contains no line through X_1 (in fact X_1 cannot even be a flecnode): its other singular points (if any) are thus (Bruce & Giblin 1981) at repeated roots of the discriminant (consider H_∞ as quadratic in x_1)

$$\Delta = \{H_\infty(\phi)\}^2 - 4x_2^2\{M_\infty(\phi)\}^2.$$

Moreover (Bruce & Giblin 1981) T_∞ is reducible iff Δ is a perfect square (or $H_\infty(\phi)$ and $x_2 M_\infty(\phi)$ have a common factor). As Δ is already a difference of two squares we have the classical equation $u^2 = v^2 + w^2$ and as we have a unique factorization domain the classical argument applies to show that if Δ is a perfect square,

$$H_\infty(\phi) = A(B^2+C^2)$$

$$x_2 M_\infty(\phi) = ABC,$$

for some A, B and C. As $H_\infty(\phi)$ has odd degree, so has A; we can take $A = x_2$; otherwise $H_\infty(\phi)$ and $M_\infty(\phi)$ would have a common factor. The same reason shows that B and C are distinct, so we are in case (i) and not (ii).

One may consider $f \mid \pi_1, f \mid \pi_2$ as functions of x_2 and x_3 by substituting $x_1 = x_2, -x_2$ respectively. All the critical points lie on the conic $\partial_3 \phi = 0$, and one could approach the classification problem in this way. The trouble is that the relation between the two functions is not very close, and we have no useful way of characterizing the condition that a critical point of $f \mid \pi_1$ and one of $f \mid \pi_2$ correspond to the same critical value.

4.4. *Lines in T_∞ and singular pencils*

The case when T_∞ contains a line of points P whose polar conics Σ_{P} form a pencil with a common vertex played a part above. We now ask when T_∞ can contain a line λ corresponding to a pencil with no common vertex. There is a unique type of such a pencil; it has Segre symbol [1; 1] see Hodge & Pedoe (1952, p. 289).

THEOREM 4.4.1. *Suppose $\lambda \subset T$ is a line of points whose polar quadrics do not have a common vertex. Then λ passes through an* E-*point of S.*

Proof. Let P be the point on λ with Σ_{P} a plane-pair. We may normalize as in §4.1 with P at W. Then

$$H(F) = w^2\{(ac-b^2)\,M(\phi)\} + w\{aH(\phi) - bcxM(\phi)\} + \{bxH(\phi) - c^2x^2M(\phi)\}.$$

The line λ is given by ratios $(x:y:z)$ for which the three terms in curly brackets vanish. Now $ac \neq b^2$, else the polar quadric of P would be a repeated plane, which cannot occur in a pencil of quadrics of the type considered. Hence $M(\phi) = 0$, so $aH(\phi) = 0$. But if $M(\phi)$ and $H(\phi)$ both vanish on λ, then by lemma 4.2.1 the pencil has a common vertex. Hence $H(\phi) \neq 0$, so $a = 0$. But now P is an E-point (the tangent plane $x = 0$ at P meets S in the line-triple $0 = x = \phi(0, y, z)$).

5. CASE OF SINGULAR PENCIL WITH COMMON VERTEX

5.1. *Review of cubic functions on $\mathbb{C}^2$*

Our procedure for the classification in the case when the net contains a singular pencil with common vertex will be to reduce to the case of cubic functions on $\mathbb{C}^2$. We therefore begin by reviewing earlier results (Wall 1979) for this case. Suppose then $\phi: \mathbb{C}^2 \to \mathbb{C}$ is a cubic function, associated to the homogeneous $\Phi: \mathbb{C}^3 \to \mathbb{C}$ as usual by $\Phi(u, v, w) = w^3\phi(u/w, v/w)$; we may also suppose that some level curves $\phi^{-1}(a)$ are elliptic (equivalently, none have any singularities at infinity).

Cases fall into three divisions, according to the type of the binary cubic $\Phi(u, v, 0)$ (corresponding to S_∞ in this paper): we have type I (distinct factors), type II (a repeated factor) and type III (perfect cube). Each type is further divided into species, according to the nature of the critical levels; we label these by small greek letters. The species may also be characterized in terms of the interrelation of S_∞ and T_∞ (intersection of $H(\Phi) = 0$ with $w = 0$).

The enumeration is conveniently made by using normal forms. For type I, we take

$$\phi(u, v) = u^3+v^3+6auv+3bu+3cv+d.$$

The species are as follows. Write

$$\Delta = 27a^8-18a^4bc-4a^2(b^3+c^3)-b^2c^2.$$

Then the details are as in table 5.1.1. Similarly for types II, III we take

$$\phi(u, v) = u^2v+v^2+bu+cv+d, \quad \text{(type II)},$$

$$\phi(u, v) = u^3+v^2+au+b, \quad \text{(type III)},$$

and we have the six species described in table 5.1.2.

TABLE 5.1.1

species	conditions	singular levels	T_∞
Iα	$\Delta \neq 0, \quad b \neq c^3$	$A_1/A_1/A_1/A_1$	1, 1, 1,
Iβ	$\Delta = 0, \quad a \neq 0, \quad b^3 \neq c^3$	$A_2/A_1/A_1$	2, 1
Iγ	$b^3 = c^3 \neq 0, \quad a^6, -27a^6$	$A_1^2/A_1/A_1$	(1), 1, 1
Iδ	$b^3 = c^3 = a^6 \neq 0$	A_2/A_1^2	(1), 2
Iε	$b^3 = c^3 = -27a^6 \neq 0$	A_3/A_1	(3)
Iζ	$b = c = 0, \quad a \neq 0$	A_1^3/A_1	(1, 1, 1)
Iη	$a = bc = 0, \quad b^3 \neq c^3$	A_2/A_2	$\underline{2}$, 1
Iθ	$a = b = c = 0$	D_4	∞

TABLE 5.1.2

species	conditions	singular levels	T_∞
IIα	$b \neq 0, \quad 4c^3+27b^2 \neq 0$	$A_1/A_1/A_1$	1, 1, 1
IIβ	$b \neq 0, \quad 4c^3+27b^2 = 0$	A_2/A_1	2, 1
IIγ	$b = 0, \quad c \neq 0$	A_1^2/A_1	(1), 1, 1
IIδ	$b = c = 0$	A_3	(3)
IIIα	$a \neq 0$	A_1/A_1	(1), 1, 1
IIIβ	$a = 0$	A_2	(1), $\underline{2}$

The notation for T_∞ denotes the three points, with their multiplicities: a point is enclosed in parentheses if it also belongs to S_∞, and underlined if its polar conic is a repeated line. It should be noted that a root of multiplicity r of a binary form is a singularity of type A_{r-1}.

5.2. *Canonical forms: reduction to study of* ϕ

Now suppose $\lambda \subset T_\infty$ is a line consisting of points P whose polar quadrics Σ_P have a common vertex V. We cannot have $V \in \lambda$, else V would be a singular point at infinity of S. By duality, each $P \in \lambda$ lies on the vertex of Σ_V which is thus a plane-pair with vertex λ, or a repeated plane.

The basic idea of the classification in this case is to choose a plane π through λ but not through V, and to consider the restriction ϕ of F to π. We shall see that each critical point Q of ϕ

determines two of F (in general distinct) on the line VQ. Then we use, first, the species of ϕ (as in (5.1) above) and, secondly, a calculation of critical values of F in terms of those of ϕ to determine when critical points (or values) coincide.

This programme requires us to list various cases arising, and use normal forms for each case. We can always take λ to be the line $x_0 = x_3 = 0$; and take V to be X_0 (if it is not on π_∞) or X_3 (if it is). We find

$$(V \equiv X_0) \quad F = F_\infty(x_1, x_2, x_3) + 3bx_3^2x_0 + 3cx_0^2x_3 + dx_0^3,$$

$$(V \equiv X_3) \quad F = \Phi(x_0, x_1, x_2) + 3b'x_0^2x_3 + 3c'x_0x_3^2 + d'x_3^3.$$

When $V \equiv X_0$ it is of little advantage to normalize further. There are three subcases: (i) (the general case) $b \neq 0$, $c \neq 0$; (ii) $b = 0$, $c \neq 0$; (iii) $b \neq 0$, $c = 0$ (here X_0 is a critical point of corank 2). We ignore $b = c = 0$ since here X_0 is the only critical point (of type $\tilde{E}_6$) of F. We shall not exclude case (iii) since this is the general case for a critical point of corank 2, and gives a convenient occasion to recall the results of Wall (1980*a*). When $V \equiv X_3$, the polar quadric Σ_V is given by $b'x_0^2 + 2c'x_0x_3 + d'x_3^2 = 0$; we cannot have $d' = 0$ (else S_∞ is singular at X_3) so may take $d' = 1$, and then (replacing $x_3 + c'x_0$ by x_3) $c' = 0$; now set $b' = -r^2$, so that Σ_V is the plane-pair $(x_3 + rx_0)(x_3 - rx_0) = 0$. There are two subcases: (iv) $r \neq 0$, and (v) $r = 0$, so we have a repeated plane, corresponding to case (vi) of §4.1.

We next choose the plane π. When $V \equiv X_3$, choose π as $x_3 = 0$ (half-way between the two parallel planes of Σ_V). Thus $\phi(x_1, x_2) = \Phi(1, x_1, x_2)$. When $V \equiv X_0$, we again choose a plane $x_3 =$ constant: it does not much matter which constant (in case (i) there is a preferred choice $x_3 = -c/b$ half-way between the planes of Σ_V, but this is not available for cases (ii), (iii)) so we take

$$\phi(x_1, x_2) = F_\infty(x_1, x_2, 1).$$

When $V \equiv X_0$, S_∞ is nonsingular iff the curve $\Phi = 0$ is so ϕ may belong to any of the species listed in (5.1); when $V \equiv X_3$ we have

$$F_\infty(x_1, x_2, x_3) = \Phi_\infty(x_1, x_2) + x_3^3,$$

so S_∞ is nonsingular iff Φ_∞ has distinct roots, i.e. ϕ is of type I.

Now consider critical points of f. Since these satisfy $0 = \partial f/\partial x_1 = \partial f/\partial x_2$, we deduce

$$(V \equiv X_3) \quad (x_1, x_2) \text{ is a critical point of } \phi,$$

$$(V \equiv X_0) \quad \text{either } x_3 = 0 \quad \text{or} \quad (x_1/x_3, x_2/x_3) \text{ is a critical point of } \phi.$$

Conversely if Q is a critical point of ϕ (on $x_3 = 0$ if $V \equiv X_3$ and on $x_3 = 1$ if $V \equiv X_0$), any critical point of the restriction of f to the line VQ is a critical point of f. Moreover, if the critical points on VQ are distinct, and not at V, they have the same type as the critical point of ϕ; if they coincide (but not at V), we have the 'sum' of Sebastiani & Thom (1971) with an A_2 singularity. This doubles multiplicities and changes types as follows:

$$A_1 \to A_2, \quad A_2 \to D_4, \quad A_3 \to E_6, \quad D_4 \to \tilde{E}_6.$$

Critical points on $x_3 = 0$ (when $V = X_0$) are investigated similarly: the conditions $\partial f/\partial x_1 = \partial f/\partial x_2 = 0$ show that the corresponding point is a repeated root of $\Phi_\infty = 0$, and the repeated root (if any) determines a line through V along which we may check $\partial f/\partial x_3$.

5.3. *Relation between critical values of ϕ and f*

Now let Q be a critical point of ϕ with critical value α. We next seek to compute the critical values of f on the line VQ. We take x_3 as parameter on this line. Then if $\mathrm{V} \equiv \mathrm{X}_3$, we have

$$f = x_3^3 - 3r^2x_3 + \alpha.$$

The critical points $x_3 = \pm r$ are on the two planes of Σ_{V}, and the critical values are $\alpha \mp 2r^3$. If $\mathrm{V} \equiv \mathrm{X}_0$,

$$f = \alpha x_3^3 + 3bx_3^2 + 3cx_3 + d,$$

and since $\Phi = 0$ is nonsingular, $\alpha \neq 0$. The critical points are given by $\alpha x_3^2 + 2bx_3 + c = 0$, and so coincide iff $b^2 = \alpha c$; the critical values are the numbers $d + \beta$ such that

$$\alpha x^3 + 3bx^2 + 3cx - \beta$$

has a repeated root, i.e. such that

$$\alpha^2\beta^2 + 4\alpha c^3 - 4\beta b^3 + 6\alpha\beta bc - 3b^2c^2 = 0. \tag{1}$$

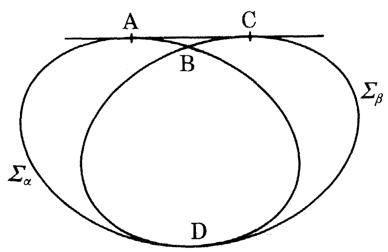

FIGURE 1. A, $\alpha = 0$; B, $\alpha = \infty$, $\beta = 0$; C, $\beta = \infty$; D, $\alpha = \infty_0$, $\beta = -c^2/b$.

For a critical point on $x_3 = 0$, suppose x_2 is the repeated factor of Φ_∞: then $0 = a_{111} = a_{112}$. On $x_3 = 0$,

$$\tfrac{1}{3}\partial f/\partial x_3 = a_{113}x_1^2 + c,$$

giving two points (distinct if $c \neq 0$), since a_{113} cannot vanish else X_1 would be singular on S_∞. The corresponding critical values both equal $d (\beta = 0)$. This may be regarded as the case of (1) with $\alpha = \infty$. In case (ii) ($b = 0$), (1) simplifies to $\beta^2 = -4c^3/\alpha$, and in case (iii) ($c = 0$), it simplifies to give $\beta = 0$ or $\beta = 4b^3/\alpha^2$.

To handle (1) in the general case, we use a standard geometrical method (see, for example, Todd 1947, p. 90). Associate to α the point $\mathrm{P}_\alpha \equiv (\alpha^2, \alpha, 1)$ on the conic Σ_α: $y^2 = xz$. Then (1) states that P_α lies on the line with dual coordinates

$$[\beta^2,\ 6bc\beta + 4c^3,\ -(4b^3\beta + 3b^2c^2)].$$

As β varies, these lines envelope a conic Σ_β

$$0 = c(3b^2Y + 4cZ)^2 + 4b^4X(2b^2Y + 3cZ).$$

Set $\alpha_0 = b^2/c$. The equation of Σ_β in point-coordinates is

$$0 = x(3\alpha_0 z - 4y) + \alpha_0(2\alpha_0 z - 3y)^2.$$

It meets the original conic thrice where $\alpha = \alpha_0 (\beta = -c^2/b^2)$ and once where $\alpha = \infty (\beta = 0)$. Three of the common tangents coincide at $\alpha = \alpha_0$; the other is at $\alpha = 0$ (and joins it to the point $\beta = \infty$).

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From each point of Σ_α, we can draw two tangents to Σ_β: they are distinct unless the point lies on Σ_β ($\alpha = \alpha_0$ or $\alpha = \infty$). The contacts give the values of β corresponding to the given values of α. Note that $\alpha = \infty$ corresponds to critical points on $x_3 = 0$ (each giving two distinct critical points of f); $\alpha = \alpha_0$ to critical points of ϕ giving coincident critical points of f (which then lie in the preferred plane $x_3 = -c/b$). While $\beta = -c^2/b$ arises only from $\alpha = \alpha_0$, $\beta = 0$ arises from $\alpha = \frac{3}{4}\alpha_0$ as well as from $\alpha = \infty$.

We can apply similar geometrical methods to the other cases too. Σ_β has four-point contact with Σ_α in case (iv), coincides with Σ_α in case (v), and is degenerate in cases (ii) and (iii).

5.4. *Enumeration*

We can now enumerate cases. Simplest is case (v) (Σ_V a repeated plane: this is the same as case (vi) of §4.2). Here, ϕ can be any affine cubic of type I; the critical points of f correspond to those of ϕ, have the same critical values, and types determined by sum with x_1^3. We thus have eight species, which we denote by R:

(Rα)	$A_2/A_2/A_2/A_2$,	(Rβ)	$D_4/A_2/A_2$,	(Rγ)	$A_2^2/A_2/A_2$,	(Rδ)	D_4/A_2^2
(Rε)	E_6/A_2,	(Rζ)	A_2^3/A_2,	(Rη)	D_4/D_4,	(Rθ)	$\tilde{E}_6$.

Next we consider case (iii) (V of corank 2). Here, ϕ can have any of the species listed in §5.1. Each critical point Q with $\phi(Q) = \alpha$ contributes to the multiplicity of V as critical point, and adds a critical point at level $4b^3/\alpha^2$. Thus V itself is of type D_4, D_5 or E_6 according as ϕ has type I, II or III, and 'coalescence' can occur if $\phi(Q_1) = -\phi(Q_2)$ giving equal critical values for f. I showed in Wall (1980*a*) that all conceivable coalescences can in fact occur, giving 28 species in all. In table 5.4.1 we denote them by D_4, D_5 or E_6 according to the type of ϕ. Here (D_4α′) and (D_4γ), for example, represent different strata of the same type. In fact (D_4α″) is also composed of two different strata. However, we can identify (Rβ) ≡ (D_4η), (Rδ) ≡ (D_4η′), (Rε) ≡ (E_6β) (Rη) ≡ (D_4θ) since the listing of cases with a critical point of corank 2 is complete.

TABLE 5.4.1

(D_4α)	$D_4/A_1/A_1/A_1/A_1$	(D_4α′)	$D_4/A_1^2/A_1/A_1$	(D_4α″)	$D_4/A_1^2/A_1^2$
(D_4β)	$D_4/A_2/A_1/A_1$	(D_4β′)	$D_4/A_2A_1/A_1$	(D_4β″)	$D_4/A_2/A_1^2$
(D_4γ)	$D_4/A_1^2/A_1/A_1$	(D_4γ′)	$D_4/A_1^3/A_1$	(D_4γ″)	$D_4/A_1^2/A_1^2$
(D_4δ)	$D_4/A_2/A_1^2$	(D_4δ′)	$D_4/A_2/A_1^2$		—
(D_4ε)	$D_4/A_3/A_1$	(D_4ε′)	D_4/A_3A_1		—
(D_4ζ)	$D_4/A_1^3/A_1$	(D_4ζ′)	D_4/A_1^4		—
(D_4η)	$D_4/A_2/A_2$	(D_4η′)	D_4/A_2^2	(D_4θ)	D_4/D_4
(D_5α)	$D_5/A_1/A_1/A_1$	(D_5α′)	$D_5/A_1^2/A_1$		—
(D_5β)	$D_5/A_2/A_1$	(D_5β′)	D_5/A_2A_1		—
(D_5γ)	$D_5/A_1^2/A_1$	(D_5γ′)	D_5/A_1^3	(D_5δ)	D_5/A_3
(E_6α)	$E_6/A_1/A_1$	(E_6α′)	E_6/A_1^2	(E_6β)	E_6/A_2

The simplest case is (ii): here each critical point of ϕ gives rise to two for f, of the same type and with distinct critical values, and $\beta = \beta'$ implies $\alpha = \alpha'$. In case (iv) we have much the same, but the extra complication that if $\alpha' = \alpha + 4$, the critical value $\beta = \alpha + 2$ appears twice, so we can have some coalescence. The same can also take place in (i); thus the cases arising under (ii) and (iv) are all included in the enumeration for (i).

Here there are two further complications. If ϕ is of type II (or III), we count an A_1 (or A_2) with '$\alpha = \infty$' giving rise to A_1^2 (or A_2^2) at the critical level $W = 0$. Any singularities of ϕ with critical level $\alpha_0 (= b^2/c)$ are 'doubled' by sum with x^3 as above. Otherwise each singularity of ϕ gives rise to two of f (at different levels), of the same type, but coalescences of critical levels may occur as above. Note, however, that by Poncelet's theorem (see, for example, Todd 1947, p. 92) 'cyclic' chains $\alpha_i \to \beta_i, \beta_{i+1}$ $(1 \leqslant i \leqslant n-1)$, $\alpha_n \to \beta_n, \beta_1$ cannot occur.

We now tabulate the possibilities: we again borrow the notation of §5.1, but add the prefix L. In table 5.4.2 we allow for singularities with critical level α_0 but not for coalescences. All cases with a critical point of corank 2 are excluded. The first row names the case, the second gives the singularities of f with $\alpha = \alpha_0$ or ∞, and the third the remaining singularities of ϕ (each of which gives two singular points of f of the same type). In the cases LIδ_2, ϵ_2, ζ_2, ζ_3 no coalescence is available. In cases LIβ_2, γ_2, γ_3, δ_1, ϵ_1, ζ_1, η and LIIβ_2, γ_2, γ_3, δ, LIIIα_2, β just one coalescence is possible (for the six latter cases, with $\alpha = \infty$); we denote the case when it occurs by adding a prime to the symbol. The remaining cases LIα_1, α_2, β_1, γ_1, LIIα_1, α_2, β_1, γ_1 and LIIIα_1 yield more possibilities, which are enumerated in table 5.4.3. In this table a symbol such as (L_1, L_2, L_3) signifies that the critical levels L_i belong to critical values α_i of ϕ, and the corresponding critical values β of f satisfy $\alpha_i \to \beta_i, \beta_{i+1}$, (so the chords $\alpha_i\alpha_{i+1}$ of Σ_α are tangent to Σ_β). Rather than include $\alpha_1 = \infty$ in the symbol, we replace it by a vertical stroke: $|L_2, L_3)$.

TABLE 5.4.2

α_1	α_2	β_1	β_2	γ_1	γ_2	γ_3	δ_1	δ_2	ϵ_1	ϵ_2	ζ_1	ζ_2	ζ_3	η
—	A_2	—	A_2	—	A_2^2	A_2	—	A_2^2	—	A_2	—	A_2^3	A_2	—
$A_1/A_1/A_1/A_1$	$A_1/A_1/A_1$	$A_2/A_1/A_1$	A_2/A_1	$A_1^2/A_1/A_1$	A_1/A_1	A_1^2/A_1	A_2/A_1^2	A_2	A_3/A_1	A_3	A_1^3/A_1	A_1	A_1^3	A_2/A_2

α_1	α_2	β_1	β_2	γ_1	γ_2	γ_3	δ
A_1^2	A_2/A_1^2	A_1^2	A_2/A_1^2	A_1^2	A_2^2/A_1^2	A_2/A_1^2	A_1^2
$A_1/A_1/A_1$	A_1/A_1	A_2/A_1	A_2	A_1^2/A_1	A_1	A_1^2	A_3

α_1	α_2	β
A_2^2	A_2/A_2^2	A_2^2
A_1/A_1	A_1	A_2

TABLE 5.4.3

X	X′	X″	X‴	X$^{\text{iv}}$	X$^{\text{v}}$	X$^{\text{vi}}$	X$^{\text{vii}}$
LIα_1	A_1, A_1, A_1, A_1	$(A_1, A_1)A_1, A_1$	$(A_1, A_1)(A_1, A_1)$	$(A_1, A_1, A_1)A_1$	(A_1, A_1, A_1, A_1)	—	—
LIα_2	A_1, A_1, A_1	$(A_1, A_1)A_1$	(A_1, A_1, A_1)	—	—	—	—
LIβ_1	A_2, A_1, A_1	$(A_2, A_1)A_1$	$A_2(A_1, A_1)$	(A_2, A_1, A_1)	(A_1, A_2, A_1)	—	—
LIγ_1	A_1^2, A_1, A_1	$(A_1^2, A_1)A_1$	$A_1^2(A_1, A_1)$	(A_1^2, A_1, A_1)	(A_1, A_1^2, A_1)	—	—
LIIα_1	A_1, A_1, A_1	$(A_1, A_1)A_1$	$\vert A_1), A_1, A_1$	$\vert A_1)(A_1, A_1)$	(A_1, A_1, A_1)	$\vert A_1, A_1)A_1$	$\vert A_1, A_1, A_1)$
LIIα_2	A_1, A_1	(A_1, A_1)	$\vert A_1)A_1$	$\vert A_1, A_1)$	—	—	—
LIIβ_1	A_2, A_1	(A_2, A_1)	$\vert A_2)A_1$	$\vert A_1)A_2$	$\vert A_2, A_1)$	$\vert A_1, A_2)$	—
LIIγ_1	A_1^2, A_1	(A_1^2, A_1)	$\vert A_1^2)A_1$	$\vert A_1)A_1^2$	$\vert A_1^2, A_1)$	$\vert A_1, A_1^2)$	—
LIIIα_1	A_1, A_1	(A_1, A_1)	$\vert A_1)A_1$	$\vert A_1, A_1)$	—	—	—

5.5. *Analysis of T_∞ and the V-relation*

Using the normal form of §5.2 for $V = V_0$ we compute

$$H_\infty(F) = cx_3 H(F_\infty) - b^2 x_3^2 M(F_\infty),$$

where, as in §4.1, $M(\phi)$ denotes $\partial_{22}\phi\,\partial_{33}\phi - (\partial_{23}\phi)^2$. If $c = 0$, x_3^2 is a repeated factor, and we have the cases discussed in Wall (1980*a*). Otherwise, we can write

$$G(x_1, x_2, x_3) = F_\infty(x_1, x_2, x_3) - \alpha_0 x_3^3,$$

with $\alpha_0 = b^2/c$, and find $H_\infty(F) = cx_3 H(G)$. We have already called attention to the significance of α_0 as critical level of ϕ: we now see that the singularities of T_∞ are determined in terms of those of $\phi = \alpha_0$ as follows. If $\phi = \alpha_0$ has a singularity of type A_2 (A_3), the tangent there is a twofold (threefold) line of T_∞. Each singularity of type A_1 gives a singularity of type A_1 of T_∞; other singularities are on $x_3 = 0$ except that if $\phi = \alpha_0$ is equianharmonic, $H(G)$ is a triangle. As to singularities on $x_3 = 0$, these are given by tables 5.1.1 and 5.1.2: we have A_1^3 if ϕ has species Iα, γ, ζ, IIα, γ or IIIα; A_3A_1 if ϕ has species Iβ, δ or IIβ, A_5 if ϕ has species Iε or IIδ, D_4 if ϕ has species Iη or IIIβ while x_3 is a repeated line for species Iθ. Singularities on $x_3 = 0$ (a pencil with common vertex) are all adjugate base points.

In particular, there is another line in T_∞ if $H_\infty(G)$ is reducible, i.e. $\phi = \alpha_0$ has type A_1^2 (only occurs in species Iγ_2, Iδ_3, IIγ_2) A_1^3 (Iζ_2) or A_3 (Iε_2, IIδ_2 – but this case was excluded) or is equianharmonic.

We now describe the V-relation. Each A_1-singularity of $\phi = \alpha_0$ gives one of $G = 0$ which is V-related to the corresponding A_2-singularity of F. On Φ_∞ we have a singular point of type A_3 for species Iβ, δ, IIβ which is V-related to both A_2-singularities of f, and for Iε and IIδ an A_5-singularity, V-related to both A_3-points. Finally for Iη and IIIβ the point P of Φ_∞ has type D_4 on T_∞; Σ_P is a repeated plane (so these cases also occur under (v)); and P is V-related to all four A_2-singularities of f. As we shall find in other examples, it is not so easy to locate the points of contact of S_∞ and T_∞ corresponding to coincidences of critical values.

Now consider $V = X_3$. Here we find

$$H(F) = x_3 H(\phi) - (x_3^2+1)\,N(\phi),$$

with $N(\phi) = \partial_{11}\phi\,\partial_{22}\phi - (\partial_{12}\phi)^2$. Taking ϕ in standard form for type I yields

$$H_\infty(F) = x_3\{x_1x_2(bx_1+cx_2) - a^2(x_1^3+x_2^3)\} - x_1x_2x_3^2 = x_3 G_\infty,$$

where G_∞ represents a nodal cubic if $a \neq 0$, a triangle if $a = 0$. As above, the intersections of G_∞ with $x_3 = 0$ are the points of $H_\infty(\Phi)$ with their due multiplicities; in this case, these are V-related to all the critical points of f (of multiplicity greater than 1), the rules being as above.

6. The E_∞-point case

6.1. *Classification*

The case when S contains an E_∞-point P arose in §3.4 and occurred again as case (iii) in §4.1: in this chapter we give a full discussion. It was shown in §1.2, and proposition 1.4.2 that for P a nonsingular point of S, equivalent characterizations are (a) the tangent plane to S at P meets S in three lines through P; (b) the polar quadric Σ_P is reducible; (c) P is singular on the Hessian surface T (or on the parabolic curve).

We saw in §2.4 that we may choose coordinates (with π_∞ as $x_0 = 0$) so that

$$F = 3x_1^2x_2 + \Phi(x_0, x_2, x_3),$$

and in the same section we discussed the nature of the critical points in $x_2 = 0$, classifying P into one of five types. The remaining critical points of F lie in $x_1 = 0$ and coincide with the critical points of ϕ. Moreover, those not on $x_2 = 0$ have the same type as the corresponding critical point of ϕ for we can then take $x_1x_2^{\frac{1}{2}}$ as local coordinate, and so obtain the singularity in standard form. The condition that F_∞ is nonsingular is here equivalent to having Φ_∞ nonsingular, i.e. with three distinct asymptotes, so of type I in the notation of Wall (1979) and §5.1.

We can picture the situation in terms of the plane $\mathbb{C}^2$ on which ϕ is defined, thus giving a pattern of level curves (see, for example the figures in Wall (1980*b*)) with the line $l(x_2 = 0)$ drawn in the plane: l must not be parallel to an asymptote of $\phi = 0$. Then $\phi_l = \phi | l$ is a cubic (not of lower degree), and so has either (*a*) two critical points A_1 (at different levels) or (*b*) one critical point of type A_2. These correspond to the types of E_∞-point as defined in §2.4.

We also refined this classification in §2.4 by writing a_r or b_r when there are r critical points of f V-related to P, or equivalently r critical points of ϕ lying on l. The other critical points of ϕ_l correspond to points of tangency (in case b_0, an inflexional tangent) of l with level curves of ϕ. Such points may also lie at critical levels of ϕ. We should thus refine the previous classification by adding a prime to denote the cases a_0', a_1', b_0' when l touches a critical level curve, or two primes a_0'' when it touches two.

For given ϕ, a general line l is thus of type a_0. Types a_0', a_1, b_0 each define one-parameter families of lines; and a_0'', a_1', a_2, b_0', b_1 each occur for a finite number (if any) of lines l. For the one-parameter families we note that the condition that l passes through a specified critical point, or touches a curve at a specified critical level, defines an irreducible family of lines l (in the latter case, since l cannot be a component of a level curve). It is not clear whether type b_0 defines an irreducible family.

Since when l is tangent (inflexional tangent) to some level curve, we have singularities A_1^2 (A_2^2) at that level in $x_2 = 0$, and when l passes through (is nodal tangent at) an A_1 singularity of ϕ, that point is a critical point of type A_3 (A_5) for f, it is now easy (§6.2) to list all cases, provided we determine the finite numbers of above possibilities for l. It is also not difficult (§6.4) to determine what happens when l passes through a singularity of ϕ of higher type.

6.2. *Enumeration*

We now enumerate the cases arising. First exclude the cases when f has a critical point of corank 2: then ϕ does not have type Iθ, and l cannot pass through singularities of ϕ of type A_2 or A_3. We describe when the various types of line l can occur, according to the type of ϕ.

A general line l has type a_0. If Γ is a singular level curve of ϕ and is irreducible, a general tangent has type a_0'; If Γ consists of a conic and a line, a general tangent to the conic has type a_0'. Recall that l does not meet Γ at infinity, so cannot be an asymptote or a component. An inflexional tangent l to a general level curve is of type b_0; to a critical level Γ is of type b_0'. If Γ is a nodal cubic, there are three inflexional tangents: there is thus (at least) one line of type b_1, unless all three points of inflexion are at infinity, i.e. ϕ has type ζ (or θ). If Γ is a cuspidal cubic, there is just one inflexional tangent; however this is an asymptote if ϕ has type δ. If Γ is reducible, it has no inflexional tangents.

Next let R be a critical point of ϕ, of type A_1, lying on the level curve Γ, and consider lines l through R. If l is tangent to Γ at R but not a component of Γ, then it has type b_1. If it passes through another critical point of type A_1 (at a different level), it has type a_2. If l touches another critical level curve Γ', it has type a_1'; otherwise type a_1. There are only finitely many lines of type a_1'; let us count how many. If Γ' is nodal, it has class 4 so we expect four tangents from R; however, if Γ is reducible, and the line m is a component, then m is an asymptote to

Γ' and an inflexional tangent at infinity, so accounts for two of the tangents, leaving two; if Γ is a triangle, there are none left. If Γ' is cuspidal it has class 3, so we expect three tangents (which may be reduced to 1); if Γ' consists of a conic and a line, the conic has class 2 so there are two tangents (there are no cases with Γ and Γ' both reducible).

We have left to last the discussion of the most difficult case, a_0''. Again we start with two singular level curves, Γ and Γ', neither of which is a triangle, and now seek to count the number of common tangents – mainly to decide when this number is zero. If Γ and Γ' are both nodal, hence of class 4, we may expect 16 common tangents (counted with multiplicities). However, each asymptote is a common tangent, and Γ and Γ' have three-point contact there. Thus it counts as three common tangents. There are now seven remaining. If the asymptote is an inflexional tangent (which occurs only when ϕ has type γ), it has even higher multiplicity, but there still remain common tangents. If Γ is nodal and Γ' cuspidal, we now expect $4\times 3-3-3-3 = 3$ common tangents (here ϕ has type β, so there are no osculating asymptotes); if Γ and Γ' are both cuspidal we have $3\times 3-3\times 3 = 0$ common tangents (type η). Otherwise suppose Γ' consists of a conic (and a line): if Γ is nodal, we expect $4\times 2-3-3 = 2$ common tangents; this is correct for type γ but for type ε the residual line of Γ' touches the conic and is an inflexional tangent to Γ, and thus accounts for both common tangents, leaving zero. Finally if Γ is cuspidal (type δ) we have $3\times 2-3-3 = 0$ common tangents. To summarize: although any two critical levels have common tangents for ϕ of type α, β or γ, the two critical levels in the remaining cases δ, ε, ζ, η have no common tangents.

We give our conclusions in table 6.2.1. In the left-hand column we describe the critical levels: these will be denoted by Γ_P, Γ_Q, Γ_R (in order), and critical points on (for example) Γ_Q will be denoted by Q, Q′, … . A tick means that the case occurs.

TABLE 6.2.1

	type of V …	a_0	a_0'	a_0''	a_1	a_1'	a_2	b_0	b_0'	b_1
type of ϕ	species									
$A_1/A_1/A_1/A_1$	α	✓	✓	✓	✓	✓	✓	✓	✓	✓
$A_2/A_1/A_1$	β	✓	Γ_P	Γ_P, Γ_Q	Q	Q, Γ_P	Q, R	✓	Γ_P	Q
	β'	—	Γ_Q	Γ_Q, Γ_R	—	Q, Γ_R	—	—	Γ_Q	—
$A_1^2/A_1/A_1$	γ	✓	Γ_P	Γ_P, Γ_Q	P	P, Γ_Q	P, Q	✓	Γ_Q	P
	γ'	—	Γ_Q	Γ_Q, Γ_R	Q	Q, Γ_P	Q, R	—	—	Q
A_2/A_1^2	δ	✓	Γ_P	—	Q	Q, Γ_P	—	✓	—	Q
A_3/A_1	ε	✓	Γ_P	—	Q	Q, Γ_P	—	✓	Γ_Q	Q
A_1^3/A_1	ζ	✓	Γ_Q	—	P	—	P, Q	✓	—	Q
A_2/A_2	η	✓	Γ_P	—	—	—	—	✓	Γ_P	—

Also $\gamma'' a_1'(Q, \Gamma_R)$, $\delta' a_0'(\Gamma_Q)$, $\varepsilon' a_0'(\Gamma_Q)$, $\zeta' a_1(Q)$.

In each case we have listed the critical points (if any) through which l passes, and the critical levels (if any) that it touches. Where there are several possibilities giving results of the same type, we have only listed one of them. The symbol describing the critical points at different critical values can be immediately derived: for V of type a_0, add A_1^2 at the two levels indicated, for type a_1, add an A_1^2 and convert the indicated A_1 to an A_3; for type a_2 convert both to type A_3. For V of type b_0, add an A_2^2 at the level indicated; for type b_1, convert the A_1 to an A_5.

6.3. *Analysis of T_∞ and the V-relation*

We now study the Hessian, the quartic curve T_∞ and the V-relation. Normalizing the equation as in §6.1 we have

$$H = x_2 H(\phi) - x_1^2 M(\phi),$$

with $M(\phi) = \partial_{00}\phi\,\partial_{33}\phi - (\partial_{03}\phi)^2$. We notice again that both F and H admit the symmetry $x_1 \leftrightarrow -x_1$.

The quartic T_∞ being of the form $x_1^2 f_2(x_2, x_3) + f_4(x_2, x_3) = 0$, we can determine its singularities. These satisfy $0 = \partial H_\infty/\partial x_1 = 2x_1 f_2(x_2, x_3)$: those on $x_1 = 0$ are at repeated roots of f_4, and those on $f_2 = 0$ must also satisfy $f_4 = 0$. In fact only when f_2 and f_4 have a common factor λ with $\lambda^2 \nmid f_2$, $\lambda^2 \nmid f_4$ is there a singular point other than X_1 not on $x_1 = 0$: we then have two singularities each of type A_1 at the intersections (other than X_1) of λ with the residual cubic. The point X_1 itself has type A_1 if f_2 has distinct roots, A_3 if $f_2 = \lambda^2$ but $\lambda \nmid f_4$, and A_5 if $f_2 = \lambda^2$, $\lambda \mid f_4$ (but $\lambda^2 \nmid f_4$). The singular points on $x_1 = 0$ are at the repeated roots of f_4: for a root of multiplicity $r \geqslant 2$ that is not a root of f_2 we have a point of type A_{r-1}; one which does lie on $f_2 = 0$ is of type A_3 if $r = 2$ and a triple point of type D_4 (D_5) if $r = 3$ ($r = 4$).

We can also describe T_∞ in each of the 25 cases that arise: for example if the highest common factor of $f_2(f_4)$ is a line λ, the residual cubic Γ is nodal (cuspidal) iff f_4/λ has a repeated factor (is a perfect cube). Further if $f_2 = \lambda^2$ then λ is an inflexional tangent of Γ whereas if f_2 has distinct factors then λ is a chord if $\lambda^2 \nmid f_4$, a tangent if $\lambda^2 \mid f_4$, $\lambda^3 \nmid f_4$, and λ passes through a singular point of Γ if $\lambda^3 \mid f_4$.

For T_∞ we have further information in that one factor of f_4 is picked out as x_2. The above classification is entirely in terms of repeated factors of $f_2 f_4 = x_2 M_\infty(\phi) H_\infty(\phi)$. We now interpret these in terms of the above picture:

$x_2 \mid M_\infty(\phi)$ iff X_1 is an E_∞-point of type (*b*) (by above); $x_2 \mid H_\infty(\phi)$ iff the polar conic of X_3 (at infinity on $x_2 = l$) is singular; $M_\infty(\phi)$ is a perfect square iff l passes through a critical point of ϕ (by above); $H_\infty(\phi)$ has a repeated factor if ϕ has species Iβ, Iδ or Iη and a threefold factor for species Iε (or Iθ) (by Wall 1979).

Finally, a common factor G of $M_\infty(\phi)$ and $H_\infty(\phi)$ gives (by lemma 4.1) a line of points whose polar quadrics have a common vertex R. This line passes through X_1, so corresponds to a point Q at infinity in the plane of ϕ. The point R lies on the vertex $x_1 = x_2 = 0$ of the polar of X_1; thus the line $l(x_2 = 0)$ passes through the vertex R of the polar conic of Q with respect to ϕ. This condition is not in general relevant to properties of f. However, if G is a repeated factor of $H_\infty(\phi)$, so Q is singular (type A_1 or A_2), then R is a critical point of ϕ (type A_2 or A_3) V-related to Q. Since $R \in l$, it gives a critical point of f of corank $\geqslant 2$. The only exception to this is when ϕ has type Iη.

We can now give a complete analysis of V-relatedness (which incidentally determines the types of quartic that can occur as T_∞). We saw already in §2.4 the various cases for the critical points (those on $x_1 = x_2 = 0$) V-related to X_1 itself.

Next, for each λ (if any), dividing $M_\infty(\phi)$ and $x_2 H_\infty(\phi)$ each to the first power only, we have two singular points of T_∞ of type A_1 on λ. We find that if $\lambda \mid H_\infty(\phi)$ these are adjugate base points and not V-related to anything. If, however, $\lambda = x_2$ they are not adjugate base points; as $x_2 \mid M_\infty(\phi)$, X_1 is an E_∞-point of type b, and as $x_2^2 \nmid M_\infty(\phi)$, X_1 is of type b_0 and is E-related to the critical points of T_∞ on x_2 (which are not adjugate base points).

we find that if $a = 0$ we can take P at X_3. If also $d = 0$, P lies on S_∞ so is an E-point. We must check that S_∞ is nonsingular: it has invariants $I = 12bc$, $J = -4(b^3+c^3)$, so is singular iff $b^3 = c^3$. The cases arising are thus in species Iα, with $a = d = 0$, which we identified as one component of the Iα″ stratum, and in Iη with $d = 0$, giving the stratum Iη′.

On the other hand we can also enumerate these cases as in §6.2 starting from the E_∞-point. Here we assume that l passes through a critical point of ϕ of type A_2 or A_3. Then f has a critical point of type D_4 (D_5, $\tilde{E}_6$) if l passes through a point of type A_2 (A_3, D_4) or E_6 if l is tangent at a cusp, for the corank is increased by 1 and the multiplicity by 2 (if the intersection number of l with the critical level is 2) or 4 (otherwise). The types arising are given in table 6.4.1. For ϕ of type θ, V is of type a_0 ($D_4\alpha''$) or b_0 ($D_4\eta'$) or else l passes through the critical point, when f has only one singular point (type $\tilde{E}_6$). The only case causing a problem of identification is $D_4/A_1^2/A_1^2$; it suffices to note that for an E_∞-point of the first type above, the restriction of f to the plane it defines has a term x_3^2 whereas for (ii), this restriction ϕ is of type θ. It is rather surprising that for ϕ of type β there are two different kinds of lines of type a_1' (tangents from the A_2-point to a critical level of type A_1).

TABLE 6.4.1

type of V ...	a_1	a_1'	a_2	b_1
species of ϕ				
β	$D_4\gamma$	$D_4\gamma'$, $D_4\zeta$	$D_4\varepsilon$	$E_6\alpha$
δ	$D_4\gamma''$	$D_4\zeta'$	$D_4\varepsilon'$	$E_6\alpha'$
ε	$D_5\gamma$	$D_5\gamma'$	$D_5\delta$	—
η	$D_4\delta$	$D_4\delta'$	$D_4\theta$	$E_6\beta$

6.5. *Overlap between enumerations of* §5 *and* §6

We now study the overlap between the classifications of §5 and of §6: namely, the case when T_∞ contains both an E_∞-point P and a line λ corresponding to a singular pencil with common vertex. We first consider the relatively commonly occurring case $P \in \lambda$. We have seen in earlier subsections of §6 (especially §6.3) that this occurs if and only if λ is a common factor of $M_\infty(\phi)$ and $H_\infty(\phi)$: the line l must go through the pole of a singular conic of the polar pencil of ϕ, corresponding to a simple (except for type η) factor of $H_\infty(\phi)$.

In general the polar pencil has three singular conics. The base points of the pencil are the four critical points, forming a complete quadrangle; the vertices of the three line-pairs are the diagonal points of the quadrangle. We observe that in specializations of this, a line through one of the three vertices and a critical point will still pass through two critical points (or one of multiplicity at least 2 – an excluded case for l, since then f will have a critical point of corank 2). Thus l cannot have type a_1, a_1' or b_1.

One also finds in most specializations that some vertices coincide with critical points of multiplicity greater than or equal to 2, so l may not pass through them. When asymptotes of ϕ do not concur ($a \neq 0$ in the notation of I) we have three vertices available for ϕ of species α, γ or ζ, one available for species β, δ but none for species ε (which will thus not occur in the lists below). When the asymptotes do concur, two of the vertices are at infinity; there remains one for species α, γ while for species ε, there is a line of vertices, so we always have a line λ in T_∞.

In the context of §5 we took the line λ as $x_3 = 0$: this corresponds to the line at infinity in the plane of ϕ. We saw that the intersections of λ with the residue of T_∞ were always given by

$H_\infty(\phi) = 0$; the points on S_∞ are given by $\Phi_\infty = 0$. Thus E_∞-points are given by common zeros of these two: we have one such point for ϕ of species Iγ, δ, ε, IIγ, δ, IIIα, β and three such points for ϕ of species Iζ, θ. We shall now show in each of these cases how to determine the type of the corresponding E_∞-point. After §6.4 we may exclude all cases when there is a critical point of corank greater than or equal to 2.

To avoid confusion, we shall write ψ here for the function denoted ϕ in §5, and retain ϕ for the same use as earlier in this section. We begin by determining the type of ϕ. Note that the polar quadric Σ_P of the E_∞-point P splits as $\pi_P \pi_\phi$, where π_P is the tangent plane at P; ϕ is the restriction of f to π_ϕ. The critical points of ϕ are also critical points of f; they are determined by the rules given in §5.2.

Now π_ϕ meets the plane $\pi(\psi)$ on which ψ is defined in a line m of symmetry. Indeed, in the various cases arising

$$\text{(I)} \quad \psi = u^3+v^3+6Auv+3B(u+v)+D,$$

$$\text{(II)} \quad \psi = u^2v+3v^3+6pv+a,$$

$$\text{(III)} \quad \psi = u^3+3v^2-3p^2u+a,$$

the E_∞-point is at infinity at $(1, -1, 0)$, $(1, 0, 0)$ or $(0, 1, 0)$ respectively so $\pi(\phi) \cap \pi(\psi)$ is given by $u = v$, $u = 0$, or $v = 0$ respectively. We see that for I, m passes through two of the four critical points of ψ; for II it passes through one and we may count another at infinity; and for III m again passes through both critical points. Thus the critical points (and values) of $\psi|m$ occur among those of ψ. These determine the critical points (and values) of ϕ by the same rules as in §5.2 – except that we have one dimension lower. Again, we may regard the cases (ii), (iv) of §5.2 as effectively special cases of (i), so need not consider them further.

For ψ of type I or III, if $\psi|m$ has distinct critical points, then the critical values and hence the species of ϕ are determined as follows:

$A_1/A_1/A_1/A_1$ (species α) in general,

$A_1/A_1^2/A_1$ (species γ) if coalescence occurs,

$A_2/A_1/A_1$ (species β) if α_0 is a critical value,

whereas if the critical points of $\psi|m$ coincide, we have

A_2/A_2 (species η) in general,

D_4 (species θ, excluded) if α_0 is a critical value.

For ψ of type II, we have to count ∞ as a critical value of m. The cases arising for ϕ are

$A_1^2/A_1/A_1$ (species γ) in general,

A_1^3/A_1 (species ζ) if coalescence occurs,

A_1^2/A_2 (species δ) if α_0 is a critical value.

We note the predicted non-occurrence of species ε.

Next we determine the type of the E_∞-point P. The tangent plane at P meets the ψ-plane in a line $(u+v = 2A,\ v = 0,\ \text{infinity})$ on which ψ has a constant value $(8A^3+6AB+D,\ a,\ \infty)$. Note that this is the common value of ψ at its two remaining critical points. Now the behaviour of $\phi|l$ is determined by this value of ψ by the same rule as in §5.3 (here the dimension is one lower still). Thus for ψ of type I or II we find that P has type b iff this value equals α_0; for type III we see by direct verification that P always has type b.

Now the enumeration in § 5.4 allowed us to deduce in each case the symbol giving the number and type of the critical levels of f. From this we can decide in each case whether P (if of type a) has type a_0 or a_2 and (by counting critical levels) to which symbol a_0, a_0', a_0'', a_2, b_0, b_0' to assign it, and hence finally place it in the table § 6.2. The cases are given in table 6.5.1. We see that almost all the cases with P of type a_0, a_2 or b_0 and ϕ of type α, β, γ, δ, ζ, η do in fact occur (the others, namely $\beta a_0'$, $\beta a_0''$, $\gamma' a_0''$, $\delta a_0'$, δb_0, could presumably be excluded by direct arguments).

Table 6.5.1

LI	γ_1^1	γ_1^2	γ_1^3	γ_1^4	γ_1^5	γ_2	γ_2'	γ_3	γ_3'	δ_1	δ_1'	δ_2
E	αa_0	$\alpha a_0'$	γa_0	$\gamma' a_0'$	$\alpha a_0''$	αb_0	γb_0	βa_0	$\beta' a_0'$	ηa_0	$\eta a_0'$	ηb_0
LI	ε_1	ε_1'	ε_2	ζ_1	ζ_1'	ζ_2	ζ_3					
E	αa_2	γa_2	βa_2	$\alpha a_0''$	$\gamma a_0''$	$\beta b_0'$	$\beta' a_0''$					
LII	γ_1^1	γ_1^2	γ_1^3	γ_1^4	γ_1^5	γ_1^6	γ_2	γ_2'	γ_3	γ_3'	δ	δ'
E	γa_0	$\gamma' a_0'$	$\gamma a_0'$	ζa_0	$\gamma a_0''$	$\zeta a_0'$	γb_0	ζb_0	δa_0	$\delta' a_0'$	$\gamma' a_2$	ζa_2
LIII	α_1^1	α_1^2	a_1^3	α_1^4	α_2	α_2'	β	β'				
E	αb_0	γb_0	$\alpha b_0'$	$\gamma b_0'$	βb_0	$\beta' b_0'$	ηb_0	$\eta b_0'$				

The E-classification here should be regarded as definitive; the fact that T becomes reducible does not always change the pattern of critical points, so it may become reducible in 'different ways' for different points in the same stratum. Hence the fact that $E\alpha a_0''$ and six other strata appear twice in the list is not contradictory.

We should also consider the cases Rγδεζθ. But as there is no critical point of corank 2, we only have Rγ and Rζ, which already appeared as LIIIβ and β′, and as $E\eta b_0$ and b_0'.

There is also the case when T contains P and λ with $P \notin \lambda$. I claim in this case however that T_∞ must also contain a line through P. For if $x_1^2 f_2 + f_4$ is reducible we either have a factor not involving x_1 (proving our assertion) or both factors are linear in x_1:

$$(x_1^2 f_2 + f_4) = (x_1 l + q)(x_1 l' + q').$$

Equating coefficients gives $lq' = -l'q$. If $l | q$, we again have a line through P. If not, we can take $l' = l$; then $q' = -q$ and we have two conics. Further analysis shows that if T_∞ does contain such a line λ, we have $f_4 = f_2 l^2$ for some linear form l so that T_∞ decomposes into four lines.

We conclude by describing the case when there is more than one E_∞-point at infinity. Suppose S is a nonsingular level surface of f. Then the line λ joining two E_∞-points either lies on S or meets S again in a third E_∞-point. As S_∞ contains no line, we see that λ contains three E_∞-points. These are singular on T: as the intersection multiplicity of λ with T exceeds 4, λ lies on T. Hence $\lambda \subset T_\infty$ defines a singular pencil with common vertex.

Following the approach of § 5.2 we choose a plane π through λ but not through the vertex and consider the restriction ϕ of f to π. Then Φ_∞ defines the three E_∞-points, whose polar quadrics, being plane-pairs, meet π in line-pairs. Thus all lie on $H(\Phi) = 0$. It follows that ϕ is of type Iζ or Iθ. Conversely, for ϕ of type Iζ or Iθ in § 5.2 we obtain a function f with three E_∞-points.

We gave enumerations in §5.4. We obtain

$$(\mathrm{D_4}\zeta)\quad \mathrm{D_4/A_1^3/A_1},\quad (\mathrm{D_4}\zeta')\quad \mathrm{D_4/A_1^4},\quad (\mathrm{D_4}\theta)\quad \mathrm{D_4/D_4},$$

$$(\mathrm{R}\zeta)\quad \mathrm{A_2^3/A_2},\quad (\mathrm{R}\theta)\quad \tilde{\mathrm{E}}_6,$$

$$(\mathrm{LI}\zeta_1)\quad \mathrm{A_1^3/A_1^3/A_1/A_1},\quad (\mathrm{LI}\zeta_1')\quad \mathrm{A_1^3/A_1^4/A_1},\quad (\mathrm{LI}\zeta_2)\quad \mathrm{A_2^3/A_1/A_1}.$$

$$(\mathrm{LI}\zeta_3)\quad \mathrm{A_2/A_1^3/A_1^3}.$$

Recall that we gave two main cases: if the common vertex V of the Σ_Q with $Q \in \lambda$ is finite we take $V \equiv X_0$ and our normal form (for type ζ) is

$$f = x_1^3 + x_2^3 + 6\lambda x_1 x_2 x_3 + a x_3^3 + 3b x_3^2 + 3c x_3 + d;$$

if V is infinite we take $V \equiv X_3$; the term $6\lambda x_1 x_2 x_3$ is replaced by $6\lambda x_1 x_2$. In either case the symmetries $x_1 \to \omega x_1$, $x_2 \to \omega^2 x_2$ $(\omega^3 = 1)$ and $x_1 \leftrightarrow x_2$ of f permute the E_∞-points which are thus of the same type. We can see (for example, by considering the E-relation) that there cannot be any more E_∞-points.

The cases with no critical point of corank 2 arise in the classification of §6.2 as follows:

$\mathrm{A_1^3/A_1^3/A_1/A_1}$	$\mathrm{A_1^3/A_1^4/A_1}$	$\mathrm{A_1^3/A_1^3/A_2}$	$\mathrm{A_2^3/A_1/A_1}$	$\mathrm{A_2^3/A_2}$
$\mathrm{E}\alpha \mathrm{a}_0''$	$\mathrm{E}\gamma \mathrm{a}_0''$	$\mathrm{E}\beta' \mathrm{a}_0''$	$\mathrm{E}\beta \mathrm{b}_0'$	$\mathrm{E}\eta \mathrm{b}_0'$

7. Functions with two critical values

7.1. *Counting methods*

Suppose $F: \mathbb{C}^3 \to \mathbb{C}$ is an affine function with just two critical values. As the total multiplicity of critical points is 8, at least one of the critical values must have multiplicity $m \geq 4$ i.e. m is the sum of multiplicities of the singular points of $F^{-1}(0)$. By the classification (Bruce & Wall 1979) (and excluding corank 2 critical points as enumerated in §5.4) we have one of the following:

$$\mathrm{A_4,\ A_3A_1,\ A_2^2,\ A_2A_1^2,\ A_1^4,\ A_5,\ A_4A_1,\ A_3A_1^2,\ A_2^2A_1,\ A_5A_1,\ A_2^3}.$$

By §1.4 if the surface has $\mathrm{A_2^2}$ singularities or worse (i.e. $\mathrm{A_5}$, $\mathrm{A_2^2A_1}$, $\mathrm{A_5A_1}$, $\mathrm{A_2^3}$), there is an E_∞-point. The singularities $\mathrm{A_4}$, $\mathrm{A_3A_1}$, $\mathrm{A_4A_1}$, $\mathrm{A_3A_1^2}$ are handled explicitly below. This leaves $\mathrm{A_2A_1^2}$, $\mathrm{A_1^4}$.

The $\mathrm{A_1^4}$ singularity is excluded as follows. If the singular points are P_0, P_1, P_2, P_3, then each of the six lines P_iP_j meets π_∞ at a point (not an E-point) where the intersection number of S_∞ with H_∞ is 2. This already accounts for all 12 intersections. Thus there can be no other pairs of points at the same critical level. If there are only two critical levels, the other must be of type $\mathrm{A_4}$. But then the pinch line meets π_∞ at a further point of intersection of S_∞ and H_∞: a contradiction.

We may try the same argument in the other cases. A critical level of type X corresponds to an intersection multiplicity $m(\mathrm{X})$ of S_∞ with H_∞, where $m(\mathrm{X}) = a(\mathrm{X}) + b(\mathrm{X})$ and $a(\mathrm{X}) = \Sigma\{(i+j)$ for each pair A_i, A_j at the level indicated$\}$ equals the multiplicity of X multiplied by one less than the number of critical points; $b(\mathrm{X}) = \Sigma\{\max 2(i-2, 0) : \mathrm{A}_i \in \mathrm{X}\}$. The cases arising are given in table 7.1.1. If we exclude the cases when there is an E_∞-point, each of these cases is associated with just one critical level, and the sum of the $m(\mathrm{X})$ corresponding to all critical levels of F cannot exceed 12.

If there is an E_∞-point of type a, then the intersection number is only 2 whereas table 7.1.1 (for A_1^2/A_1^2, A_1^2/A_3 or A_3/A_3) would lead us to expect 4; E_∞-points of type b do not cause exceptions. The cases when there is more than one E_∞-point were enumerated in §6.5: only one case A_2^3/A_2 has just two critical levels. So even in the presence of an E_∞-point of type a, we have $\Sigma m(X) \leqslant 14$.

TABLE 7.1.1

X	A_1	A_2	A_1^2	A_3	A_1A_2	A_1^3	A_4	A_3A_1	$A_2A_1^2$	A_1^4	A_4A_1	$A_3A_1^2$
$a(X)$	0	0	2	0	3	6	0	4	8	12	5	10
$b(X)$	0	0	0	2	0	0	4	2	0	0	4	2
$m(X)$	0	0	2	2	3	6	4	6	8	12	9	12

The possibilities for just two critical levels (and no E_∞-points) are thus as follows:

$$A_4A_1/A_3,\ A_4A_1/A_2A_1,\ A_4/A_4,\ A_4/A_3A_1,\ A_4/A_2A_1^2,\ A_3A_1/A_3A_1.$$

We shall see that in fact each of these cases occurs, essentially uniquely.

It is also easy to enumerate the cases $\Sigma\, m(X) \leqslant 14$, but as this allows several cases that do not in fact occur, we content ourselves with referring back to §6.

In §§7.2–7.4 we go over these lists more carefully. As in Wall (1980*a*) each case determines f essentially uniquely, and hence the elliptic cubic S_∞, and we shall calculate the j-invariants of these curves.

7.2. *The* E_∞-*point case*

It is easy to extract from §6.2 a list of those cases where there are just two critical values:

$E\delta a_1'$	$E\delta b_1$	$E\varepsilon a_1'$	$E\varepsilon b_0'$	$E\varepsilon b_1$	$E\zeta a_2$	$E\zeta b_1$	$E\eta b_0'$
$A_2A_1^2/A_3A_1$	A_5A_1/A_2	$A_3A_1^2/A_3$	$A_3/A_2^2A_1$	A_3/A_5	$A_3A_1^2/A_3$	A_1^3/A_5	A_2^3/A_2.

One may observe that the lists of §5 only give one case with just two critical values: the final case above.

Note that the methods of §7.1 give an alternative proof that cases $\delta a_0''$, $\varepsilon a_0''$, $\zeta a_0''$, $\eta a_0''$ do not occur. For these would yield functions of types A_3A_1/A_1^3, $A_2A_1^2/A_1^4$, A_1^5/A_1^3, $A_2A_1^2/A_2A_1^2$, with $m(X)$ equalling 18, 20, 26, 16, each an impossibility.

We can give equations for these cases by using the normal form of Wall (1980*a*) or §5.1 for ϕ and then determining the line l.

It suffices to indicate how the calculations can be performed. In *species* δ we use an alternative normal form (cf. Wall 1980*b*) in which the critical points are all real; thus we take

$$g = u^3 - 3uv^2 + 3(u^2+v^2) + 3u,$$

with $\quad P(-1, 0)\quad$ of type $\quad A_2,\ g(P) = -1$

and $\quad Q_\epsilon(1, 2\epsilon)\quad$ of type $\quad A_1,\ g(Q_\epsilon) = 7, \quad \epsilon = \pm 1.$

The line $L_\lambda : v-2 = \lambda(u-1)$ through Q_1 admits -1 as a critical value of $g|L_\lambda$ iff $0 = (\lambda-1)^3 \times (9\lambda-5)$. Here $\lambda = 1$ gives the line PQ_1, so $\lambda = \frac{5}{9}$ gives a line of type a_1'. The Hessian form at Q_1 is $6u^2-12uv$: here $u = 1$ is the line Q_1Q_2, so $u-1 = 2\ (v-2)$ is of type b_1.

In *species* ε, we may take $a = 1$ in the normal form of table 5.1.1. The Hessian form at Q (determining the nodal tangents, type b_1) is $9u^2-6uv+9v^2$; if L_λ is the line $v+3 = \lambda(u+3)$ through Q then the critical points of g on L_λ are Q and $u = -(3\lambda^3-6\lambda^2+4\lambda-3)/(\lambda^3+1)$; the condition that the corresponding critical value equals -10 reduces to $0 = (\lambda-1)^4(11\lambda^2-10\lambda+11)$. Again $\lambda = 1$ corresponds to PQ: the other two factors give lines of type a_1'.

The inflexional tangents of Γ_Q can be found as follows. Substitute $u = x-3$, $v = y-3$ to reduce the equation to

$$x^3+y^3 = 9x^2-6xy+9y^2,$$

with parametrization $(9t^2-6t+9,\ 9t^3-6t^2+9t,\ t^3+1)$. The line (X, Y, Z) is an inflexional tangent if, for some α, β

$$9Y+Z = \beta^3, \quad 9X-6Y = 3\alpha,$$

$$9X+Z = \alpha^3, \quad 9Y-6X = 3\alpha^2,$$

whence $5X = 3\alpha\beta^2+2\alpha^2\beta$, $5Y = 2\alpha\beta^2+3\alpha^2\beta$ and

$$5(\alpha^3-\beta^3) = 45(X-Y) = 9(\alpha\beta^2-\alpha^2\beta),$$

giving $5\alpha^2+14\alpha\beta+5\beta^2 = 0$. Take $\alpha = 1-\sqrt{6}$, $\beta = 1+\sqrt{6}$ and substitute in the equations to obtain $-X = 5+\sqrt{6}$, $-Y = 5-\sqrt{6}$, $Z = 64$. Thus the inflexional tangents of Γ_Q are $(5+\sqrt{6})x+(5-\sqrt{6})y = 64$; their product is thus parallel to $19x^2+62xy+19y^2$.

In *species* ζ we can choose $v = u$ as the line of type a_2 joining the nodes; the tangents at the node, type b_1, are $u = 0$ and $v = 0$.

Finally in *case* η, $u = -1$ is an inflexional tangent (b_0').

Once the equation of l is determined we have

$$F = 3w^2 l(u, v) + g(u, v).$$

Hence
$$F_\infty = 3w^2 l_\infty(u, v) + u^3+v^3.$$

The tangents from W to S_∞ are thus the lines

$$l_\infty(u, v)(u^3+v^3) = 0$$

and if $l_\infty(u, v) = \alpha u+\beta v$, this quartic has invariants $I = 12\alpha\beta$, $J = 4(\alpha^3+\beta^3)$. Hence $j = -4\alpha^3\beta^3/(\alpha^3-\beta^3)^2$. If we are given $ll' = pu^2+quv+pv^2$ then we have $l' = \beta u+\alpha v$, $p = \alpha\beta$, $q = \alpha^2+\beta^2$, $j = -4p^3/(q^3-3pq^2-2p^3)$. For δ, we used a different normalization, and now need the cross ratio of the four values of u/v: 0, $\sqrt{3}$, $-\sqrt{3}$ and (a_1') $\frac{9}{5}$ or (b_1) 2.

A short calculation now yields values of j as follows:

	$E\delta a_1'$	$E\delta b_1$	$E\epsilon a_1'$	$E\epsilon b_0'$	$E\epsilon b_1$	$E\zeta a_2$	$E\zeta b_1$	$E\eta b_0'$
j	$2(53/9)^3$	$125/4$	$(11/2)^3$	$-\frac{1}{2}(19/27)^3$	$27/2$	1	0	0

7.3. *The* A_4 *case*

The equation of a surface with an A_4 singularity can always be taken in the form

$$0 = wxy+x^2z+yz^2+dy^3.$$

Here there is a singularity at W, of type A_4; also if $d = 0$ one at Y, of type A_1; but no more. The cases $d \neq 0$ are all equivalent.

For the affine case, we must choose a 'plane at infinity' not through W: we can take it as $w = ax+by+cz$; then the affine equation is obtained by substituting

$$w = ax+by+cz+1.$$

Thus $F(x, y, z) = dy^3+x^2z+yz^2+xy(ax+by+cz+1)$.

We now investigate the critical points of F. Now

$$\partial F/\partial z = x^2 + 2yz + cxy$$

vanishes on a cone parametrized by

$$(x, y, z) = \lambda(2\theta, -1, 2\theta^2 - c\theta)$$

(the line $x = y = 0$ corresponds to $\theta = \infty$). Substituting these values gives

$$y\frac{\partial F}{\partial y} - x\frac{\partial F}{\partial x} = \lambda^3 \delta(\theta), \quad \frac{\partial F}{\partial x} = \lambda^2 f(\theta) - \lambda,$$

where
$$\delta(\theta) = -20\theta^4 + 12c\theta^3 + (4a - c^2)\theta^2 + 2b\theta - 3d,$$
$$f(\theta) = 8\theta^3 - 6c\theta^2 - (4a - c^2)\theta + b.$$

Thus the critical points other than W are given by the four roots of $\delta(\theta) = 0$: we can then calculate $\lambda = (f(\theta))^{-1}$ and thrice the critical value equals $xy = -2\theta\lambda^2$.

For $d = 0$ we must have $b \neq 0$ (as the plane at infinity cannot go through the singular point Y): the simple root $\theta = 0$ of $\delta(\theta) = 0$ corresponds to Y; the others behave as above.

The following lemma saves much calculation later.

LEMMA 7.3.1. *If $\theta_0 \neq 0$ is a root of multiplicity r of $\delta = 0$ corresponding to the critical value v_0 of F, then θ_0 is a root of multiplicity at least r of*

$$g(\theta) \equiv \{f(\theta)\}^2 + 2\theta v_0^{-1} = 0.$$

Proof. One easily verifies that

$$\delta'(\theta) = 2f(\theta) - 4\theta f'(\theta). \tag{*}$$

Now since $v_0 = -2\theta\lambda^2$ and $\lambda = \{f(\theta)\}^{-1}$ we have

$$\{f(\theta)\}^2 + 2\theta v_0^{-1} = 0,$$

proving the lemma for $r = 1$. Now observe that

$$g(\theta) - \theta g'(\theta) = \{f(\theta)\}^2 - 2\theta f(\theta) f'(\theta) = \tfrac{1}{2} f(\theta)\, \delta'(\theta).$$

We have $0 = \delta(\theta) = g(\theta)$ and $\theta \neq 0$; thus $\delta'(\theta) = 0$ implies $g'(\theta) = 0$ proving the lemma for $r = 2$. Similarly, by differentiating this identity,

$$-\theta g''(\theta) = \tfrac{1}{2}\{f(\theta)\,\delta''(\theta) + f'(\theta)\,\delta'(\theta)\},$$
$$-\theta g'''(\theta) - g''(\theta) = \tfrac{1}{2}\{f(\theta)\,\delta'''(\theta) + 2f'(\theta)\,\delta''(\theta) + f''(\theta)\,\delta'(\theta)\},$$

so that if $\delta''(\theta)$ (or $\delta''(\theta)$ and $\delta'''(\theta)$) vanish, it follows that $g''(\theta)$ (or $g''(\theta)$ and $g'''(\theta)$) vanish. Since $r \leqslant 4$, this proves the lemma.

COROLLARY 7.3.2. *If all roots of $\delta(\theta) = 0$ correspond to the same critical value v_0 of F, then $\delta(\theta)$ divides $g(\theta)$. If $d = 0$, and all nonzero roots of δ correspond to the same v_0, then $\theta^{-1}\delta(\theta)$ divides $g(\theta)$.*

Note that this corollary is applicable precisely in the cases when F has just two critical values (the other value $F = 0$ corresponding to W and, perhaps, Y).

To simplify the calculations, write $-e = 4a - c^2$, $k = 6c$. Thus

$$\delta(\theta) = -20\theta^4 + 2k\theta^3 - e\theta^2 + 2b\theta - 3d,$$
$$f(\theta) = 8\theta^3 - k\theta^2 + e\theta + b.$$

A long division leads to

$$75f(\theta)^2+\delta(\theta)\,(240\theta^2-36k\theta+25b^2/d)$$
$$=(3k^2+960e-500b^2/d)\,\theta^4+(1680b-114ek+50b^2k/d)\,\theta^3$$
$$+(75e^2-222bk-720d-25b^2e/d)\,\theta^2+(150be+108dk+50b^3/d)\,\theta.$$

The coefficients on the left are determined so as to remove the coefficients of θ^6, θ^5 and 1 on the right. Thus $\delta(\theta)$ divides $g(\theta)$ iff

$$3k^2+960e-500b^2/d = 0, \quad 1680b-114ek+50b^2k/d = 0,$$
$$75e^2-222bk-720d-25b^2e/d = 0, \quad 150be+108dk+50b^3/d = 150v^{-1}a.$$

We regard the fourth of these as an equation for v_0^{-1}. The others yield, in turn,

$$b = \frac{(180ek-3k^3)}{16800}, \qquad \frac{bk}{e^2} = \frac{x(6-x)}{56},$$
$$d = \frac{500(180ek-3k^3)^2}{(16800)^2(3k^2+960e)}, \qquad \frac{d}{e^2} = \frac{5x(6-x)^2}{2^6\times 3\times 7^2(x+32)},$$

and setting $x = k^2/10e$ and simplifying considerably gives

$$0 = 117x^3+3468x^2-26000x+28224$$
$$= (3x-4)\,(39x-196)\,(x+36).$$

Each value of x gives a solution, unique up to the multiplicative action

$$(a, b, c, d, e, k)\,\lambda = (a\lambda^2, b\lambda^3, c\lambda, d\lambda^4, e\lambda^2, k\lambda).$$

For $x = \frac{4}{3}$ we obtain

$$k = 40, \quad e = 120, \quad b = 40, \quad d = \tfrac{20}{3},$$
$$\delta(\theta) = -20(\theta-1)^4, \quad F \text{ of type } A_4/A_4.$$

For $x = \frac{196}{39}$ we obtain

$$k = 140, \quad e = 390, \quad b = 95, \quad d = \tfrac{125}{12}$$
$$\delta(\theta) = -20(\theta-\tfrac{1}{2})^3(\theta-\tfrac{25}{2}), \quad F \text{ of type } A_4/A_3A_1.$$

For $x = -36$ we obtain

$$k = 60, \quad e = -10, \quad b = -45, \quad d = \tfrac{3375}{4},$$
$$\delta(\theta) = -20(\theta-\tfrac{9}{2})^2(\theta^2+3\theta+\tfrac{25}{4}), \quad F \text{ of type } A_4/A_2A_1^2.$$

The case $d = 0$ is treated similarly. We have

$$20\{5f(\theta)+2\theta^{-1}\delta(\theta)\}^2+(k^2-810b\theta^{-1})\,\delta(\theta)$$
$$= (2k^3-120ek+16200b)\,\theta^3+(180e^2-1980bk-ek^2)\,\theta^2+(2bk^2+1890eb)\,\theta.$$

Hence for equal critical values,

$$0 = 2k^3-120ek+16200b = 180e^2-1980bk-ek^2.$$

Eliminating b gives

$$0 = 22k^4-1410ek^2+16200e^2$$
$$= (2k^2-30e)\,(11k^2-540e).$$

Thus we have two further cases:

$$k = 15, \quad e = 15, \quad b = \tfrac{5}{4}, \quad \delta(\theta) = -20\theta(\theta-\tfrac{1}{2})^3, \quad \text{with } F$$

thus of type A_4A_1/A_3,

$$k = 90, \quad e = 165, \quad b = 20, \quad \delta(\theta) = -20\theta(\theta-8)(\theta-\tfrac{1}{2})^2,$$

with F of type A_4A_1/A_2A_1.

Observe that we know from previous work (see, for example, §5.4) that critical points of corank 2 cannot occur in conjunction with an A_4, so once we know the multiplicities, the type of F is determined. We state the conclusion formally as

THEOREM 7.3.3. *Suppose F has an A_4 singularity and just two critical values. Then with the above normalization, F is uniquely determined up to the $\mathbb{C}^\times$-action in each case. There are five cases:*

type of F	d	$k(=6c)$	b	$e(=c^2-4a)$	j
A_4/A_4	$\frac{20}{3}$	40	40	120	-4
A_4/A_3A_1	$\frac{125}{12}$	140	95	390	50
$A_4/A_2A_1^2$	$\frac{3375}{4}$	60	-45	-10	$2^35^2/3^6$
A_4A_1/A_3	0	15	$\frac{5}{4}$	15	$-25/2$
A_4A_1/A_2A_1	0	90	20	165	$17^35^2/2^{10}$

Proof. It remains to determine the modulus of the elliptic curve at infinity. This has equation

$$0 = dy^3+x^2z+yz^2+xy(ax+by+cz).$$

The point (0, 0, 1) lies on this; the line $x = ty$ through it meets the curve where

$$(at^2+bt+\mathrm{d})y^3+(t^2+ct)y^2z+yz^2 = 0,$$

so is a tangent iff $$0 = t^4+2ct^3+(c^2-4a)t^2-4bt-4d.$$

The normalized coefficients of this quartic are

$$(1, \tfrac{1}{2}c = \tfrac{1}{12}k, \tfrac{1}{6}(c^2-4a) = \tfrac{1}{6}e, -b, -4d)$$

yielding, in the five cases above,

$$(1, \tfrac{10}{3}, 20, -40, -\tfrac{80}{3}). \quad (1, \tfrac{35}{3}, 65, -95, -\tfrac{125}{3}),$$
$$(1, 5, -\tfrac{5}{3}, 45, -3375), \quad (1, \tfrac{5}{4}, \tfrac{5}{2}, -\tfrac{5}{4}, 0)$$

and $$(1, \tfrac{15}{2}, \tfrac{55}{2}, -20, 0).$$

We can simplify by replacing t by a convenient multiple rt and multiplying the whole equation by some constant s. Taking $(r, s) = (2, \frac{3}{80}), (1, \frac{3}{5}), (3, \frac{1}{135}), (1, \frac{4}{5}), (1, \frac{2}{5})$ yields $(\frac{3}{5}, 1, 3, -3, -1)$, $(\frac{3}{5}, 7, 39, -57, -25)$, $(\frac{3}{5}, 1, -\frac{1}{9}, 1, -25)$, $(\frac{4}{5}, 1, 2, -1, 0)$ and $(\frac{2}{5}, 3, 11, -8, 0)$. Now for each quintuple (a, b, c, d, e) we compute in turn $I = ae-4bd+3c^2$, $J = ace+2bcd-ad^2-eb^2-c^3$, $(I/3)^3-J^2$ and $j = I^3/(I^3-27J^2)$. We find

$I =$	$2^6\times\frac{3}{5}$,	$2^{11}\times 3$,	$-2^9/3^3$,	2^4,	$3^3\times 17$,
$J =$	$-2^8/5$,	$-2^{16}\times\frac{7}{5}$,	$2^{12}\times 23/3^6\times 5$,	$-2^6/5$,	$-3^3\times\frac{349}{5}$,
$(I/3)^3-J^2 =$	$-2^{16}/5^3$,	$2^{32}/5^2$,	$-2^{24}/3^6\times 5^2$,	$-2^{13}/3^3\times 5^2$,	$2^{10}3^6/5^2$,
$j =$	-4,	50,	$2^35^2/3^6$,	$-\frac{25}{2}$,	$17^35^2/2^{10}$.

7.4. *The* A_3A_1 *case*

The arguments in this section are very similar to those in the preceding one. We have the projective normal form

$$0 = wxy + xz^2 + yz^2 + dy^3,$$

with an A_3 singularity at W, an A_1 singularity at X and, if $d = 0$, a further A_1 singularity at Y. The point Z is an E-point, with tangent plane $x+y = 0$. If $d = 0$, $(0, 1, -1, 0)$ is another E_∞-point, with tangent plane $w = 0$.

We take as 'plane at infinity' $w = ax+by+cz$: this does not pass through W. For it to avoid X we need $a \neq 0$; if $d = 0$ we also need $b \neq 0$. We may also suppose $c \neq 0$ (the case when Z lies at infinity being adequately covered by §6). We now study

$$F = dy^3 + (x+y)\,z^2 + xy(ax+by+cz+1).$$

We have
$$\partial F/\partial z = 2z(x+y) + cxy,$$

which vanishes on the cone parametrized by

$$(x, y, z) = \lambda\{(1+\theta), (1-\theta), \tfrac{1}{4}c(\theta^2-1)\}.$$

Substituting from this gives

$$y\,\frac{\partial F}{\partial y} - x\,\frac{\partial F}{\partial x} = \lambda^3(1-\theta)\,\delta(\theta), \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = \lambda^2 f(\theta) + 2\lambda,$$

where
$$\delta(\theta) = \tfrac{1}{8}\{c^2(\theta^2-1)\,(\theta^2+\theta)\} - a(1+\theta)^2 + b(1-\theta^2) + 3d(1-\theta)^2,$$

$$f(\theta) = \tfrac{1}{8}\{c^2(\theta^2-1)\,(\theta^2+3)\} + a(1+\theta)\,(3-\theta) + b(1-\theta)\,(3+\theta) + 3d(1-\theta)^2.$$

Of the eight critical points of F, four are located at W and X; the others are determined by $\delta(\theta) = 0$ with λ determined as $-2/f(\theta)$. The corresponding critical value is $v = xy = \lambda^2(1-\theta^2)$. These values are distinct from zero unless $F = 0$ acquires an extra singularity which occurs only when $d = 0$. This point is $(0, -b^{-1}, 0)$ as $\theta = -1$, $\lambda = -\tfrac{1}{2}b^{-1}$. In the general case,

$$0 = g_v(\theta) \equiv \{f(\theta)\}^2 - 4v^{-1}(1-\theta^2).$$

Here, as in §7.3, we need a lemma.

Lemma 7.4.1. *If* $\theta_0 \neq -1$ *is a root of multiplicity* r *of* $\delta(\theta) = 0$, *corresponding to the critical value* v_0 *of* F, *then* θ_0 *is a root of multiplicity at least* r *of* $g_{v_0}(\theta) = 0$.

Proof. We first verify that

$$\theta f(\theta) + (1-\theta^2) f'(\theta) \equiv (\theta-2)\,\delta(\theta) + (\theta-\theta^2)\,\delta'(\theta).$$

Now $g'(\theta) = 2f(\theta)f'(\theta) + 8v^{-1}\theta$, so

$$\begin{aligned} 2\theta g(\theta) + (1-\theta^2)\,g'(\theta) &= 2\theta\{f(\theta)\}^2 + 2(1-\theta^2) f(\theta) f'(\theta) \\ &= 2f(\theta)\{(\theta-2)\,\delta(\theta) + (\theta-\theta^2)\,\delta'(\theta)\}. \end{aligned}$$

We have seen that $\delta(\theta_0) = g(\theta_0) = 0$. Provided $1-\theta_0^2$ does not vanish we see by differentiating the above identity that if all the derivatives of δ up to the rth vanish at θ_0, the same holds for g, as required. Now $\theta_0 \neq -1$ by hypothesis, and $\theta = 1$ is not a root of δ since $a \neq 0$.

COROLLARY 7.4.2. *If all roots of* δ *correspond to the same critical value* v *of* F, *then* $\delta(\theta)$ *divides* $g_v(\theta)$. *If* $d = 0$ *and all roots other than* $\theta = -1$ *correspond to the same* v, *then* $(1+\theta)^{-1}\delta(\theta)$ *divides* $g_v(\theta)$.

To find functions with two critical values we thus need conditions on a, b, c, d, for existence of v such that $\delta(\theta)$ divides $g_v(\theta) = \{f(\theta)\}^2 - 4v^{-1}(1-\theta^2)$. Since δ, f are homogeneous of degree 1 in a, b, c^2 and d, and $c \neq 0$, we can take $c^2 = 8$. Now

$$\begin{aligned}\{f(\theta)-\delta(\theta)\}^2 - \theta^2\delta(\theta) + 7\theta\delta(\theta) &= \theta^4(15-7a+5b-3d) + \theta^3(6+11a-23b+27d)\\ &\quad + \theta^2(-24+19a+7b-45d+16a^2-16ab+4b^2)\\ &\quad + \theta(-6-23a+23b+21d+32a^2-8b^2)\\ &\quad + (9-24a-12b+16a^2+16ab+4b^2).\end{aligned}$$

We remove the term in θ^4 by subtracting $(15-7a+5b-3d)\delta$. This gives a cubic $A\theta^3+B\theta^2+C\theta+D$, and our conditions are

$$0 = A = C = B+D,$$

$$A = 9-18a+28b-30d,$$

$$C = 9-30a+28b+18d+32a^2-8b^2+(2a+6d)(15-7a+5b-3d),$$

$$A+B+D = -9+6a-28b-18d+32a^2+8b^2+(2a-6d)(15-7a+5b-3d).$$

Thus

$$\begin{aligned}0 &= A+B+C+D\\ &= -24a+64a^2+4a(15-7a+5b-3d)\\ &= 4a(9+9a+5b-3d).\end{aligned}$$

Since $a \neq 0$, $0 = 9+9a+5b-3d$. Solving this with $A = 0$ gives

$$a = -(81+22b)/108, \quad d = (27+38b)/36,$$

whence substituting gives

$$0 = 1024b^2+14526b+9477 = (512b+351)(2b+27).$$

We thus have two solutions:

(i) $b = -\frac{351}{512}$, $a = -\frac{625}{1024}$, $d = \frac{27}{1024}$, $c^2 = 8$, giving $\delta(\theta) = (\theta+\frac{1}{4})^4$, and F of type A_3A_1/A_4;

(ii) $b = -\frac{27}{2}$, $a = 2$, $d = -\frac{27}{2}$, $c^2 = 8$, with $\delta(\theta) = (\theta-2)^3(\theta+7)$, and F of type A_3A_1/A_3A_1.

For $d = 0$ we write $\delta(\theta) = (\theta+1)\epsilon(\theta)$ with

$$\epsilon(\theta) = \theta(\theta^2-1) - a(1+\theta) + b(1-\theta),$$

$$f(\theta) = \theta\epsilon(\theta) + 3\phi(\theta),$$

with

$$\phi(\theta) = (\theta^2-1) + a(1+\theta) + b(1-\theta),$$

and now

$$\begin{aligned}\{\phi(\theta)\}^2 - \epsilon(\theta)\{\theta+2(a-b)\} &= \theta^2(-1+3a+3b+a^2-2ab+b^2)\\ &\quad + \theta(a-b+4a^2-4b^2) + (1-2a-2b+3a^2-2ab+3b^2).\end{aligned}$$

For this to be a multiple of $1-\theta^2$ we need

$$0 = a-b+4a^2-4b^2 = a+b+4a^2-4ab+4b^2.$$

If $a = b$, the second equation reduces to $0 = 2a+4a^2$ so $a = 0$ or $-\frac{1}{2}$. If not, then $1+4a+4b = 0$.

Subtracting $a+b$ times this from the second equation leads to $ab = 0$. As $ab = 0$ is excluded when $d = 0$, we have just the one solution:

(iii) $a = b = -\frac{1}{2}$, $c^2 = 8$, $d = 0$, with $\epsilon(\theta) = \theta^3$, and F of type $A_3A_1^2/A_3$.

Of the three cases above, (i) appeared in §7.3; and in (iii), as $a = b$, we have an E-point (the original $(0, 1, -1, 0)$) at infinity, so this case must be among those in §7.2.

THEOREM 7.4.3. *With the above normalizations, there are just three cases (each essentially unique) where F has only two critical values. These are given by*

type of F	a	b	c^2	d	j
A_3A_1/A_4	$-\frac{625}{1024}$	$-\frac{351}{512}$	8	$\frac{27}{1024}$	50
A_3A_1/A_3A_1	2	$-\frac{27}{2}$	8	$-\frac{27}{2}$	$-17^3/2^7$
$A_3A_1^2/A_3$	$-\frac{1}{2}$	$-\frac{1}{2}$	8	0	$(\frac{11}{2})^3$

Proof. It remains to compute j. The projective cubic has equation

$$0 = dy^3+(x+y)z^2+xy(ax+by+cz),$$

so the line $x = ty$ through the point Z on the cubic is a tangent iff

$$(at^2+bt+d)y^2+ctyz+(t+1)z^2 = 0$$

has a repeated root, i.e. iff

$$4(at^2+bt+d)(t+1)-c^2t^2 = 0.$$

This quartic has normalized coefficients $(0, a, \frac{1}{6}(4a+4b-c^2), b+d, 4d)$ which are, in the above cases, $(0, -\frac{625}{1024}, -\frac{2250}{1024}, -\frac{675}{1024}, \frac{27}{256})$, $(0, 2, -9, -27, -54)$ and $(0, -\frac{1}{2}, -2, -\frac{1}{2}, 0)$. Again we replace t by rt and multiply by s, where

$$(r, s) = (\tfrac{3}{5}, -\tfrac{1024}{27}),\ (3, -\tfrac{1}{27}) \text{ and } (1, -2) \text{ thus obtaining}$$

$$\begin{array}{llll} (0, 5, 30, 15, -4) & \text{with } I = 2^5\times 3\times 5^2, & J = -5^2\times 7\times 2^7, & j = 50 \\ (0, -2, 3, 3, 2) & \text{with } I = 51, & J = -71, & j = -17^3/2^7 \\ (0, 1, 4, 1, 0) & \text{with } I = 44, & J = -56, & j = (\tfrac{11}{2})^3. \end{array}$$

COROLLARY 7.4.4. *The final case above is of type* $E\varepsilon a_1'$.

This is distinguished from the other $A_3A_1^2/A_3$ case (namely $E\zeta a_2$) by the j-invariant.

7.5. *Summary and conclusions*

Collecting together the above lists, we have the 24 cases given in table 7.5.1. for functions with two critical values.

TABLE 7.5.1

label	$D_4\delta'$	$D_3\varepsilon'$	$D_4\zeta'$	$D_4\eta'$	$D_4\theta$	$D_5\beta'$
critical levels	$D_4/A_2A_1^2$	D_4/A_3A_1	D_4/A_1^4	D_4/A_2^2	D_4/D_4	D_5/A_2A_1
	-8	2	$\frac{125}{4}$*	0*	0*	-1
label	$D_5\gamma'$	$D_5\delta$	$E_6\alpha'$	$E_6\beta$	$E\delta a_i'$	$E\delta b_1$
critical levels	D_5/A_1^3	D_5/A_3	E_6/A_1^2	E_6/A_2	$A_2A_1^2/A_3A_1$	A_5A_1/A_2
j	$\frac{125}{27}$*	1*	1*	0*	$2.53^3/3^6$	$\frac{125}{4}$*
label	$E\varepsilon a_1'$	$E\varepsilon b_0'$	$E\varepsilon b_1$	$E\zeta a_2$	$E\zeta b_1$	$E\eta b_0'$
critical levels	$A_3A_1^2/A_3$	$A_2^2A_1/A_3$	A_5/A_3	$A_3A_1^2/A_3$	A_5/A_1^3	A_2^3/A_2
j	$(\frac{11}{2})^3$*	$-19^3/2.3^9$	$\frac{27}{2}$	1*	0*	0*
critical levels	A_4/A_4	A_4/A_3A_1	$A_4/A_2A_1^2$	A_4A_1/A_3	A_4A_1/A_2A_1	A_3A_1/A_3A_1
	-4	50	$\frac{200}{729}$	$-\frac{25}{2}$	$5^217^3/2^{10}$	$-17^3/2^7$

The cases in table 7.5.1 marked with an asterisk give elliptic curves with complex multiplication. The only thing the values all have in common is being rational!

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The comments of Wall (1980*a*) about strata in the unfolding space of an $\tilde{E}_6$ singularity apply equally here, but we now have a complete list. Note that the two strata of type $A_3A_1^2/A_3$ are distinguished by the value of *j*.

After the listing of cases in which *F* has only two critical levels, we can now determine all the possibilities for *F*. We shall only attempt this in the sense of determining the types of critical point and whether they occur at the same critical level, though a similar but more complicated argument should suffice to determine the possible strata (connected components of the space of functions *F* with critical points of a given type).

In the unfoldings of simple singularities, the possible types of function were determined by Lyashko (1976):

$$A_1 \to A_1 \quad A_2 \to A_2,\ A_1/A_1 \quad A_3 \to A_3,\ A_2/A_1,\ A_1^2/A_1,\ A_1/A_1/A_1.$$

The lists for A_4 and A_5 are longer: indeed, any combination of critical points of type A_i at various levels (at least two) with total multiplicity 4 (5) occurs in the unfolding of A_4 (A_5) *except* A_1^3/A_1 ($A_2A_1^2/A_1$, A_1^4/A_1, A_1^3/A_2).

Next we recall that if F_1 occurs in the versal unfolding, *U* of a function F_0, then *U* is also a versal unfolding of F_1, and so contains functions of all types occurring in the standard versal unfolding of F_1. Of course, apart from F_1 itself each will have more critical levels than F_1. Now the result we shall establish is as follows.

THEOREM 7.5.2. *Each function occurring in the versal unfolding U of a singularity of type* $\tilde{E}_6$ *occurs also in the unfolding of some* F_1, *where* F_1 *occurs in U and has two critical levels.*

The types of functions F_1 are as in table 7.5.1. The author has not succeeded in obtaining an interpretation of this list by graphs.

Proof. We divide this into three cases.

(*a*) *F* has a critical point of corank 2. Here a complete list of types of function *F* was given in Wall (1980*a*, p. 2) and the result is evident by inspection.

(*b*) *F* has no critical level of multiplicity greater than 3. Thus the critical levels allowed are A_1, A_2, A_1^2, A_3, A_2A_1, A_1^3. I claim that every combination of these with total multiplicity 8 occurs as an unfolding of one of the above. This is a matter of routine verification – there are 51 cases, 26 come from A_4/A_4, eleven more from A_4/A_3A_1, leaving $A_1^3/A_1^3/A_1/A_1$, $A_1^3/A_1^2/A_1/A_1/A_1$ and twelve cases where there are three critical levels, of multiplicities 3, 3 and 2 (including $A_1^3/A_1^3/A_1^2$, giving both the above). Of these twelve; ten come from A_5/A_3 or A_5/A_1^3, $A_2A_1/A_2A_1/A_2$ from A_5A_1/A_2 and $A_3/A_2A_1/A_1^2$ from A_4A_1/A_2A_1.

(*c*) If there is a critical level of multiplicity 5, it has type A_5, A_4A_1, $A_3A_1^2$ or $A_2^2A_1$ and we choose F_1 to have its other critical level A_3 or A_2A_1. Each of A_2/A_1, A_1^2/A_1, $A_1/A_1/A_1$ occurs in the unfolding of this, and hence *F* occurs in the unfolding of F_1.

Otherwise there is one critical level (L) of multiplicity 4, with others of lower multiplicity (W):

L can be any of A_4, A_3A_1, A_2^2, $A_2A_1^2$, A_1^4 and W any of A_3/A_1, A_2A_1/A_1, A_1^3/A_1, A_2/A_2, A_2/A_1^2, A_1^2/A_1^2, $A_2/A_1/A_1$, $A_1^2/A_1/A_1$, $A_1/A_1/A_1/A_1$.

If W does not have type A_1^3/A_1 then $A_4 \to W$, so if L is A_4, A_3A_1 or $A_2A_1^2$, $L/A_4 \to L/W$. Moreover, $A_3A_1 \to A_1^3/A_1$ so $L/A_3A_1 \to L/A_1^3/A_1$.

If L is A_2^2, we have an E_∞-point. Also if L is A_1^4, the critical level L already corresponds to intersection multiplicity 12 of S_∞ with T_∞, so if there is more, we have an E_∞-point. This occurs for each W except A_2/A_2, $A_2/A_1/A_1$, $A_1/A_1/A_1/A_1$, which appear in the unfolding of A_1^4/D_4.

We think of the E_∞-point cases, as before, in terms of a cubic function ϕ on $\mathbb{C}^2$ and a line l in $\mathbb{C}^2$. When L has type A_2^2, l is an inflexional tangent to a noncritical level of ϕ. Since the space of inflexional tangents to level curves is connected (and does not become disconnected by removing tangents at infinity, or at nodes) we can deform l through such lines to be an inflexional tangent to a critical level curve (of type A_1 or A_2 – ϕ clearly cannot be of species Iθ). This corresponds to an F with a critical level of multiplicity greater than or equal to 5, which has already been discussed.

When L has type A_1^4, l is a tangent to a critical level of ϕ of type A_1^2. (Thus ϕ has species Iγ or Iδ.) We can deform l through tangents to the conic till it passes through another critical point of ϕ (no need to meet ϕ_∞ on the way). We then have F_1 of type $A_1^4/A_1^3/A_1$ or A_1^4/D_4 and unfolding to F.

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